STABLY REFLEXIVE MODULES AND A LEMMA OF KNUDSEN

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Abstract. In his fundamental work on the stack $\bar{\mathscr{M}}_{g,n}$ of stable n-pointed genus g curves, Finn F. Knudsen introduced the concept of a stably reflexive module in order to prove a key technical lemma. We propose an alternative definition and generalise the results in his appendix to [20]. Then we give a 'coordinate free' generalisation of his lemma, generalise a construction used in Knudsen's proof concerning versal families of pointed algebras, and show that Knudsen's stabilisation construction works for plane curve singularities. In addition we prove approximation theorems generalising Cohen-Macaulay approximation with stably reflexive modules in flat families. The generalisation is not covered (even in the closed fibres) by the Auslander-Buchweitz axioms.

1. Introduction

In order to establish the stabilisation morphism from the universal curve $\bar{\mathscr{C}}_{g,n}$ to the stack $\bar{\mathscr{M}}_{g,n+1}$ Knudsen developed in [20] a theory of what he called stably reflexive modules with respect to a flat ring homomorphism. In this article we take a closer look at this theory and some of its applications both in approximation theory of modules and in deformation theory.

We reformulate Knudsen's theory by first defining an absolute notion of a stably reflexive module over a ring. In the noetherian case this is the same as a module of Gorenstein dimension 0. Then we define a stably reflexive module with respect to a flat ring homomorphism as a flat family of stably reflexive modules. By cohomology-and-base-change theory this definition is equivalent to Knudsen's.

We introduce a descending series of additive categories of modules, called *n-stably reflexive modules*. For $n = 1$ these are the reflexive modules. The stably reflexive modules are n-stably reflexive for all n. We also define an absolute and a relative notion of *n-stably reflexive complexes*. An n-stably reflexive complex $(E^{\cdot}, d^{\cdot})$ gives an n-stably reflexive module $\operatorname{coker} d^{-1}$. In the noetherian case the stably reflexive complexes are Tate resolutions relative to a base ring. Finally, there is a third concept of a flat family of *n-orthogonal modules* defined by a one-sided cohomology condition. An n-orthogonal module determines and is determined by an n-stably reflexive module through the $(n+1)$-syzygy. Together Propositions 3.5, 4.2 and 4.4 extend Theorem 2 in [20, Appendix]; see Remark 4.6.

Axiomatic Cohen-Macaulay approximation was introduced by M. Auslander and R.-O. Buchweitz in [5]. This theory has recently been defined in terms of fibred categories as to give approximation results for various classes of flat families of modules; see [18]. In Theorem 5.3 we extend the approximation results of Auslander and Buchweitz in the classical case of modules of finite Gorenstein dimension to the relative setting: Let $h\colon S \to R$ be a faithfully flat finite type algebra of noetherian

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rings and let N be an S-flat finite R-module. If N is stably reflexive w.r.t. h, respectively has finite projective dimension as R-module, then N belongs to the module subcategory X_h, respectively $\hat{\mathsf{P}}^{\mathrm{fl}}_h$. If some syzygy $\mathrm{Syz}_i^R N$ is stably reflexive w.r.t. h then N belongs to the category Y_h. For any N in Y_h there are short exact sequences of R-modules

$$0 \to L \longrightarrow M \longrightarrow N \to 0 \qquad \text{and} \qquad 0 \to N \longrightarrow L' \longrightarrow M' \to 0 \tag{1.0.1}$$

with M and M' in X_h and L and L' in $\hat{\mathsf{P}}^{\mathrm{fl}}_h$. The map $M \to N$ is a right X_h-approximation and the map $N \to L'$ is a left $\hat{\mathsf{P}}^{\mathrm{fl}}_h$-approximation. In particular X_h is contravariantly finite in Y_h and $\hat{\mathsf{P}}^{\mathrm{fl}}_h$ is covariantly finite in Y_h. In the local case there are minimal (and hence unique) approximations (1.0.1). The association $N \mapsto X$ for X equal to M and L' induce functors on the stable quotient categories. They are adjoint to the natural inclusions. Moreover, the approximations and the functors are well behaved w.r.t. base change $S \to S'$; see Theorem 5.3. In Theorems 5.1 and 5.2 we give analogous results for larger classes of modules N by using n-stably reflexive modules in the approximations. The category $\hat{\mathsf{P}}^{\mathrm{fl}}_h$ is replaced by the category $\hat{\mathsf{P}}^{\mathrm{fl}}(s)_h$ of modules of projective dimension less than or equal to s and Y_h is replaced by categories depending on n and s which properly contain Y_h. Note that these categories don't satisfy the axioms of Auslander and Buchweitz. We refer to the introduction of [18] for a discussion and further references on Cohen-Macaulay approximation.

The stabilisation morphism maps a stable n-pointed curve C with an extra section Δ to a stable $(n+1)$-pointed curve C^s [20, 2.4]. Knudsen's lemma is applied in the critical case where Δ hits a node in order to obtain flatness, functoriality of the construction of C^s, and the existence of a functorial lifting of Δ. Theorem 6.1 is our generalisation of Knudsen's lemma [20, 2.2]. The setting is a pointed deformation $S \to R \to S$ of a 1-dimensional Gorenstein ring $R/\mathfrak{m}_S R$ defined over an arbitrary field. Then Theorem 6.1 says that the ideal I defining the point; $R/I \cong S$, as module is stably reflexive w.r.t. $S \to R$ and $\mathrm{Hom}_R(I,R)/R$ is isomorphic to S. By the general results on stably reflexive modules these properties are preserved by base change. In particular these two results imply the functoriality of the stabilisation construction; see Section 7. Theorem 6.1 also says that $\mathrm{Hom}_R(I,R)$ is isomorphic to the fractional ideal $\{f \in K(R) \mid f{\cdot}I \subseteq R\}$, and determines the image of $I \otimes \mathrm{Hom}_R(I,R) \to R$. Knudsen applies the former in his proof of the key lemma; see [20, 2.2] and [21], and the latter in the construction of the clutching morphisms [20, 3.7].

In order to establish flatness of C^s, Knudsen applies deformation theory. For an ordinary double point he finds an explicit formally versal formal family of pointed local rings for which he analyses his construction of C^s; see his recent account [21]. In Theorem 6.5 we show quite generally that the 'square' of a versal family for deformations of a ring together with the 'diagonal' gives a versal family for deformations of the pointed ring. In Corollary 6.7 we calculate the versal family for a pointed isolated complete intersection singularity. In Proposition 6.9 we extend Knudsen's stabilisation construction to any flat family of plane curve singularities and obtain the relevant features. In the last section we explain the main steps in the proof of the stabilisation morphism [20, 2.4] and how these results apply.

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I thank Finn F. Knudsen who suggested to me, many years ago, to study his Lemma 2.2 in [20]. In the summer 2009 he urged me again and this time I wrote up roughly the first half of this article. I believe none of us looked much more at it until Finn went to Ann Arbor in January 2011 where he took on his own independent,

explicit approach, resulting in [21]. Encouraged, I completed my generalisation of the lemma. Finn sent me versions of his manuscript and the explicit and elegant construction of the hull inspired me to find the general Theorem 6.5 and Corollary 6.7.

2. Preliminaries

All rings are unital and commutuative. If A is a ring let Mod_A denote the category of A-modules and mod_A the category of finite A-modules.

2.1. **Coherent modules.** We follow [6, Chap. 1, §2, Exc. 11-12] and [12, Chap. 2]. Let A be a ring. An A-module N is coherent if N is finite and if all finite submodules of N are finitely presented. A ring is coherent if it is coherent as a module over itself. All finitely presented modules are coherent if and only if A is coherent. A polynomial ring over a noetherian ring is coherent. A coherent ring divided by a finitely generated ideal is coherent. Let coh_A denote the full subcategory of Mod_A of coherent modules. It's an abelian category closed under tensor products and internal Hom. Assume A is coherent. Then any coherent module has a resolution by finite free modules. Hence if M and N are coherent then $\mathrm{Tor}_i^A(M,N)$ and $\mathrm{Ext}_A^i(M,N)$ are coherent A-modules for all i. Moreover $\mathcal{S}^{-1}M$ is a coherent $\mathcal{S}^{-1}A$-module for any multiplicatively closed subset $\mathcal{S}$ of A.

2.2. **Base change.** The main tool for reducing properties to the fibres in a flat family will be the base change theorem. We follow the quite elementary and general approach of A. Ogus and G. Bergman [22].

Definition 2.1. Let $h\colon S\to R$ be a ring homomorphism and I an S-module. Let F be an S-linear functor of some additive subcategory of Mod_S containing S to Mod_R. Then the *exchange map e_I for F* is defined as the R-linear map $e_I\colon F(S)\otimes_S I\to F(I)$ given by $\xi\otimes u\mapsto F(u)(\xi)$ where we consider u as the multiplication map $u\colon S\to I$. Let $\mathfrak{m}\text{-}\mathrm{Spec}\,R$ denote the set of closed points in $\mathrm{Spec}\,R$.

Proposition 2.2 ([22, 5.1-2])**.** *Let $h\colon S\to R$ be a ring homomorphism with S noetherian. Suppose $\{F^q\colon \mathsf{mod}_S\to\mathsf{mod}_R\}_{q\geqslant 0}$ is an h-linear cohomological δ-functor.*

(i) *If the exchange map $e^q_{S/\mathfrak{n}}\colon F^q(S)\otimes_S S/\mathfrak{n}\to F^q(S/\mathfrak{n})$ is surjective for all $\mathfrak{n}$ in $Z=\mathrm{im}\{\mathfrak{m}\text{-}\mathrm{Spec}\,R\to\mathrm{Spec}\,S\}$, then $e^q_I\colon F^q(S)\otimes_S I\to F^q(I)$ is an isomorphism for all I in mod_S.*
(ii) *If $e^q_{S/\mathfrak{n}}$ is surjective for all $\mathfrak{n}$ in Z, then e^{q-1}_I is an isomorphism for all I in mod_S if and only if $F^q(S)$ is S-flat.*

Note that if the F^q in addition extend to functors of all S-modules $F^q\colon\mathsf{Mod}_S\to\mathsf{Mod}_R$ which commute with filtered direct limits, then the conclusions are valid for all I in Mod_S.

Example 2.3. Suppose R is coherent. Let $K^*\colon K^0\to K^1\to\dots$ be a complex of S-flat and coherent R-modules. Define $F^q\colon\mathsf{mod}_S\to\mathsf{coh}_R$ by $F^q(I)=\mathrm{H}^q(K^*\otimes_S I)$. Then $\{F^q\}_{q\geqslant 0}$ is an h-linear cohomological δ-functor which extends to all S-modules and commutes with direct limits.

Assume in addition that $S\to R$ is a flat and local homomorphism of noetherian rings and let (F) be a sequence in the maximal ideal $\mathfrak{m}_R$. Applying the Koszul complex it follows that (F) is a regular sequence and $R/(F)$ is S-flat if and only if the image of (F) in $R/\mathfrak{m}_S R$ is a regular sequence.

Example 2.4. Suppose R is coherent. Let M and N be coherent R-modules with N S-flat. Then the functors $F^q\colon\mathsf{mod}_S\to\mathsf{coh}_R$ defined by $F^q(I)=\mathrm{Ext}^q_R(M,N\otimes_S I)$ give an h-linear cohomological δ-functor which extends to all S-modules and commutes with direct limits.

Let $S \to R$ and $S \to S'$ be ring homomorphisms, M an R-module, $R' = R \otimes_S S'$ and N' an R'-module. Then there is a change of rings spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{Ext}^q_{R'}(\mathrm{Tor}^S_p(M, S'), N') \Rightarrow \mathrm{Ext}^{p+q}_R(M, N')\,. \tag{2.4.1}$$

In addition to the isomorphism $\mathrm{Hom}_{R'}(M \otimes_S S', N') \cong \mathrm{Hom}_R(M, N')$ there are edge maps $\mathrm{Ext}^q_{R'}(M \otimes_S S', N') \to \mathrm{Ext}^q_R(M, N')$ for $q > 0$ which are isomorphisms too if M (or S') is S-flat. If I' is an S'-module we can compose the exchange map $e^q_{I'}$ (regarding I' as S-module) with the inverse of this edge map for $N' = N \otimes_S I'$ and obtain the *base change map* $c^q_{I'}$ of R'-modules

$$c^q_{I'} \colon \mathrm{Ext}^q_R(M, N) \otimes_S I' \to \mathrm{Ext}^q_{R'}(M \otimes_S S', N \otimes_S I')\,. \tag{2.4.2}$$

We will use the following geometric notation. Suppose $S \to R$ is a ring homomorphism, M is a R-module and s is a point in $\mathrm{Spec}\, S$ with residue field $k(s)$. Then M_s denotes the fibre $M \otimes_S k(s)$ of M at s with its natural $R_s = R \otimes_S k(s)$-module structure. Now Proposition 2.2 implies the following:

Corollary 2.5. *Let $S \to R$ and $S \to S'$ be ring homomorphisms with S noetherian and R coherent. Suppose M and N are coherent R-modules, $Z = \mathrm{im}\{\mathfrak{m}\text{-}\mathrm{Spec}\, R \to \mathrm{Spec}\, S\}$ and q is an integer. Assume that M and N are S-flat.*

(i) *If $\mathrm{Ext}^{q+1}_{R_s}(M_s, N_s) = 0$ for all s in Z, then $c^q_{I'}$ in (2.4.2) is an isomorphism for all S'-modules I'.*
(ii) *If in addition $\mathrm{Ext}^{q-1}_{R_s}(M_s, N_s) = 0$ for all $s \in Z$, then $\mathrm{Ext}^q_R(M, N)$ is S-flat.*

2.3. **Reflexive modules.** Let A be a ring and M an A-module. The dual of M is the A-module $\mathrm{Hom}_A(M, A)$ which we denote by $M^\vee$. There is a natural map $\sigma_M \colon M \to M^{\vee\vee}$ given by $\sigma_M(m) = \mathrm{ev}_m$, the evaluation map. The module is called *torsionless* if σ_M is injective and *reflexive* if σ_M is an isomorphism; cf. [7, 1.4] for finite modules and noetherian rings. Suppose $G \xrightarrow{d} F \to M \to 0$ is a projective presentation of M. Define $D(M)$, the *transpose of* M, to be the cokernel of the dual $d^\vee \colon F^\vee \to G^\vee$. Put $\mathrm{Syz}\, M = \mathrm{im}\, d$ and $\mathrm{Syz}\, D(M) = \mathrm{im}\, d^\vee$.

Lemma 2.6 (cf. [7, 1.4.21])**.** *There are canonical isomorphisms*

$$\ker \sigma_M \cong \mathrm{Ext}^1_A(D(M), A) \quad \textit{and} \quad \mathrm{coker}\, \sigma_M \cong \mathrm{Ext}^2_A(D(M), A).$$

Proof. The projective presentation induces a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & (\mathrm{Syz}(DM))^\vee \arrow[r] & F \arrow[r, "\psi"] & M^{\vee\vee} & \\ 0 \arrow[r] & \mathrm{Syz} M \arrow[r] \arrow[u, "\rho"] & F \arrow[r] \arrow[u, equal] & M \arrow[r] \arrow[u, "\sigma_M"'] & 0 \end{tikzcd} \tag{2.6.1}$$

where ρ is the inclusion $\mathrm{im}\, d \subseteq (\mathrm{im}\, d^\vee)^\vee = (\mathrm{Syz}(DM))^\vee$. Then $\mathrm{coker}\, \sigma_M \cong \mathrm{coker}\, \psi \cong \mathrm{Ext}^1_A(\mathrm{Syz}(DM), A) \cong \mathrm{Ext}^2_A(D(M), A)$ and $\ker \sigma_M \cong \mathrm{coker}\, \rho \cong \mathrm{Ext}^1_A(D(M), A)$. □

Assume A is a coherent ring and recall the stable category $\underline{\mathsf{coh}}_A$ defined by A. Heller in [16]. It has the same objects as coh_A and morphisms $\underline{\mathrm{Hom}}_A(M, N)$ given as the quotient $\mathrm{Hom}_A(M, N)/\!\sim$ of the A-linear homomorphisms by the following equivalence relation: Maps f and g in $\mathrm{Hom}_A(M, N)$ are stably equivalent if $f - g$ factors through a coherent projective A-module.

In the stable category the syzygy can be made into a functor as follows. Fix a projective presentation $G \xrightarrow{d} F \to M \to 0$ for each coherent A-module M, i.e. a presentation where G and F are coherent and projective A-modules. The (first) syzygy module $\mathrm{Syz}^A M$ of M is defined to be $\mathrm{im}\, d$. Inductively define $\mathrm{Syz}^A_n M = \mathrm{Syz}^A(\mathrm{Syz}^A_{n-1} M)$ and put $\mathrm{Syz}^A_0 M = M$. As the syzygy only depends

on the choices made up to stable equivalence Syz^A induces an endo-functor on $\underline{\mathsf{coh}}_A$. One has $\underline{\mathrm{Hom}}_A(\mathrm{Syz}^A_n M, N) \cong \mathrm{Ext}^n_A(M, N)$ for all $n > 0$. Similarly D defines a duality on $\underline{\mathsf{coh}}_A$. In the following we will allow $D(M)$ and $\mathrm{Syz}^A_n(M)$ to denote any representative for the corresponding stable isomorphism class.

Given a short exact sequence $\xi\colon M_1 \to M_2 \to M_3$ in coh_A there is an exact sequence of projective presentations of ξ. After dualisation the serpent lemma gives an exact sequence in coh_A:

$$0 \to M_3^\vee \to M_2^\vee \to M_1^\vee \to D(M_3) \to D(M_2) \to D(M_1) \to 0 \tag{2.6.2}$$

Lemma 2.7. *Suppose A is a coherent ring and $\xi\colon M_1 \to M_2 \to M_3$ is a short exact sequence of coherent A-modules. Assume that the natural map $\mathrm{Ext}^1_A(M_3, A) \to \mathrm{Ext}^1_A(M_2, A)$ is injective. Then $M_1^\vee \to D(M_3)$ in* (2.6.2) *is the zero map and the short exact sequence $D(\xi)$ induces the exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathrm{Ext}^1_A(D(M_1), A) \longrightarrow \mathrm{Ext}^1_A(D(M_2), A) \longrightarrow \mathrm{Ext}^1_A(D(M_3), A) \longrightarrow \\ \mathrm{Ext}^2_A(D(M_1), A) \longrightarrow \mathrm{Ext}^2_A(D(M_2), A) \longrightarrow \mathrm{Ext}^2_A(D(M_3), A)\,. \end{aligned}$$

The last map is surjective if $\mathrm{Ext}^1_A(M_1^\vee, A) \to \mathrm{Ext}^1_A(M_2^\vee, A)$ is injective.

Proof. The connecting map $M_1^\vee \to D(M_3)$ factors through the connecting map $M_1^\vee \to \mathrm{Ext}^1_A(M_3, A)$. Since $\mathrm{Ext}^1_A(M_3, A) \to D(M_3)$ is injective $D(\xi)$ is a short exact sequence if and only if the connecting $M_1^\vee \to \mathrm{Ext}^1_A(M_3, A)$ is the 0-map, which is true by assumption. Moreover, the dual of ξ gives a short exact sequence $\xi^\vee$. Dualising once more gives a left exact sequence $\xi^{\vee\vee}$ and a natural map of complexes $\sigma_\xi\colon \xi \to \xi^{\vee\vee}$. Then the 6 terms from the long exact sequence of $\mathrm{Hom}_A(-, A)$ applied to $D(\xi)$ is naturally identified with the exact sequence obtained from the serpent lemma of σ_ξ. □

Example 2.8. Let A be a Cohen-Macaulay ring (in particular noetherian) with a canonical module ω_A; cf. [7, 3.3.16]. For an A-module M let M^v denote the A-module $\mathrm{Hom}_A(M, \omega_A)$. By local duality theory the evaluation map $M \to M^{vv}$ is an isomorphism if M is a (finite) maximal Cohen-Macaulay (MCM) module. If A is Gorenstein then A is a canonical module for A and hence MCM A-modules are reflexive.

3. n-stably reflexive modules

We define an n-stably reflexive module with respect to a flat ring homomorphism. Knudsen's definition of a stably reflexive module in Theorem 2 of [20, Appendix] implies ours. Proposition 3.5 gives the converse.

Definition 3.1. Let A be a ring and n a positive integer. An A-module M is *n-stably reflexive* if M is reflexive and $\mathrm{Ext}^i_A(M, A) = 0 = \mathrm{Ext}^i_A(M^\vee, A)$ for all $0 < i < n$. If M is n-stably reflexive for all n it's called a *stably reflexive* module.

A 1-stably reflexive module is the same as a reflexive module. If M is reflexive so is $M^\vee$. Hence $M^\vee$ is n-stably reflexive if M is.

Remark 3.2. Auslander and M. Bridger introduced *Gorenstein dimension* in [4]. For noetherian rings and finite modules what we here call a stably reflexive module is the same as a module of Gorenstein dimension 0; see [4, 3.8].

Example 3.3. As noted in Example 2.8 if A is Gorenstein (in particular noetherian) then a finite MCM A-module M is reflexive. By local duality theory $\mathrm{Ext}^i_A(M, A) = 0$ for all $i > 0$. Since $M^\vee$ also is MCM, M is stably reflexive

as an A-module. The converse is also true: Localising at a prime ideal $\mathfrak{p}$ in A, Grothendieck's local duality theorem with $\omega_{A_\mathfrak{p}} = A_\mathfrak{p}$ and $\mathfrak{m} = \mathfrak{p}A_\mathfrak{p}$ gives

$$\mathrm{H}^i_\mathfrak{m}(M_\mathfrak{p}) \cong \mathrm{Hom}_{A_\mathfrak{p}}(\mathrm{Ext}^{n-i}_{A_\mathfrak{p}}(M_\mathfrak{p}, A_\mathfrak{p}), E(k)) \text{ for all } i \tag{3.3.1}$$

where $E(k)$ is the injective hull of the residue field k and $n = \dim A_\mathfrak{p}$ ([7, 3.5.9]). Hence $\mathrm{H}^i_\mathfrak{m}(M_\mathfrak{p}) = 0$ for $i < \dim A_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ in A and M is a maximal Cohen-Macaulay A-module.

We are now going to define a relative version of the above notion.

Definition 3.4. Let n be a positive integer. Suppose $h\colon S \to R$ is a flat ring homomorphism and M an R-module. Then M is *n-stably reflexive with respect to h* if M is S-flat and the fibre $M_s = M\otimes_S k(s)$ is n-stably reflexive as an R_s-module for all $s \in \mathrm{im}\{\mathfrak{m}\text{-Spec}\, R \to \mathrm{Spec}\, S\}$. The module M is stably reflexive with respect to h if it's n-stably reflexive with respect to h for all n.

We will also say that M is n-stably (stably) reflexive *over S*.

Proposition 3.5. *Let $h\colon S \to R$ be a flat ring homomorphism and M an S-flat coherent R-module. Suppose S is noetherian, R is coherent and $n > 1$. Assume the module M is n-stably reflexive with respect to h. Then:*

(i) *For any ring homomorphism $S \to S'$ the base change $M' = M\otimes_S S'$ is n-stably reflexive with respect to $h' = h\otimes_S S'$ and as $R' = R\otimes_S S'$-module.*
(ii) *The dual $M^\vee = \mathrm{Hom}_R(M, R)$ is n-stably reflexive with respect to h.*
(iii) *For any ring homomorphism $S \to S'$ the natural map $M^\vee\otimes_S S' \to (M')^\vee$ is an isomorphism.*

Proof. Choose a projective presentation $G \to F \to M \to 0$ of M. Note that $D(M)\otimes_S S'$ is (stably) isomorphic to $D(M')$. Define the cohomological δ-functor $\{F^q\}_{q\geqslant 0}$ by $F^q(I) = \mathrm{H}^q((F^\vee \to G^\vee)\otimes_S I)$ for all S-modules I as in Example 2.3. Then $F^0(S) = M^\vee$ and $F^1(S) = D(M)$. Since $\mathrm{Ext}^1_{R_s}(M_s, R_s) = 0$ for all $s \in Z := \mathrm{im}\{\mathfrak{m}\text{-Spec}\, R \to \mathrm{Spec}\, S\}$ the functor F^0 admits base change by Corollary 2.5. Since F^1 admits base change $D(M)$ is S-flat too by Proposition 2.2. Then the base change map $c^i_{I'}\colon \mathrm{Ext}^i_R(D(M), R)\otimes_S I' \to \mathrm{Ext}^i_{R'}(D(M'), R\otimes_S I')$ is well defined. Lemma 2.6 and Corollary 2.5 imply that $\sigma_{M'}\colon M' \to (M')^{\vee\vee}$ is injective (surjective) if $\sigma_{M_s}\colon M_s \to M_s^{\vee\vee}$ is injective (surjective) for all $s \in Z$.

Moreover, since $n > 1$, Corollary 2.5 implies that the base change map

$$c^i_{I'}(M)\colon \mathrm{Ext}^i_R(M, R)\otimes_S I' \longrightarrow \mathrm{Ext}^i_{R'}(M', R'\otimes_{S'} I') \tag{3.5.1}$$

is an isomorphism for all $i < n$. In particular $M^\vee\otimes_S S' \cong \mathrm{Hom}_{R'}(M', R') = (M')^\vee$ and $M^\vee$ is S-flat. Hence the base change map $c^i_{I'}(M^\vee)$ is defined and by Corollary 2.5 an isomorphism for all $i < n$. □

Remark 3.6. Knudsen's definition in Theorem 2 (1) in [20, Appendix] assumes that R is noetherian. In this case we observe that our Definition 3.4 is equivalent to Knudsen's definition of a stably reflexive module. It says that a finite R-module M is a stably reflexive module with respect to S if for all S-modules I the exchange map $\mathrm{Hom}_R(M, R)\otimes_S I \to \mathrm{Hom}_R(M, R\otimes_S I)$ and the composition

$$M\otimes_S I \to \mathrm{Hom}_R(M^\vee, R)\otimes_S I \to \mathrm{Hom}_R(M^\vee, R\otimes_S I) \tag{3.6.1}$$

are isomorphisms, and $\mathrm{Ext}^i_R(M, R\otimes_S I) = 0 = \mathrm{Ext}^i_R(M^\vee, R\otimes_S I)$ for all $i > 0$. Assume this. By (3.6.1) M is reflexive and by Corollary 2.5 M and $M^\vee$ are S-flat. Composing with the edge isomorphisms in (2.4.1) it follows that M is stably reflexive with respect to h in the sense of Definition 3.4. By Proposition 3.5 the inverse implication is true as well.

Corollary 3.7. *Let $h\colon S \to R$ be a flat homomorphism of noetherian rings with finite Krull dimension. Let M be an S-flat finite R-module. Suppose R_s is Gorenstein for all s in $Z = \operatorname{im}\{\mathfrak{m}\text{-Spec}\, R \to \operatorname{Spec} S\}$. Then the following statements are equivalent*:

(i) *M is stably reflexive with respect to h.*
(ii) *M_s is a maximal Cohen-Macaulay R_s-module for all $s \in Z$.*
(iii) $\operatorname{Ext}^i_{R_s}(M_s, R_s) = 0$ *for all* $0 < i \leqslant \dim R_s$ *and* $s \in Z$.

Proof. (i)$\Leftrightarrow$(ii)$\Rightarrow$(iii) follows from Proposition 3.5 and Example 3.3.

(iii)$\Rightarrow$(ii) Passing to the localisation (R', M') of (R_s, M_s) at a prime ideal $\mathfrak{p}$ in R_s, (3.3.1) gives $\mathrm{H}^i_{\mathfrak{p}R'}(M') = 0$ for $i < \dim R'$ and M' is a MCM R'-module. $\square$

The following lemma generalises Lemma and Corollary 3 in [20, Appendix].

Lemma 3.8. *Let $S \to R$ be a flat ring homomorphism and $\xi\colon {}^1M \to {}^2M \to {}^3M$ a short exact sequence of coherent R-modules. Suppose R is coherent and S is noetherian.*

(i) *If ${}^1M$ and ${}^3M$ are n-stably reflexive over S, so is ${}^2M$.*
(ii) *If ${}^2M$ and ${}^3M$ are n-stably reflexive over S and $n > 1$ then ${}^1M$ is $(n-1)$-stably reflexive over S.*

Proof. In (i) and (ii) all three modules are S-flat. Let $s \in \operatorname{im}\{\mathfrak{m}\text{-Spec}\, R \to \operatorname{Spec} S\}$. As $A = R_s$ is obtained by first localising and then dividing by a coherent ideal, it's a coherent ring and similarly for the modules. For (ii), since $\operatorname{Ext}^1_A({}^3M_s, A) = 0$ Lemma 2.6 and 2.7 give that ${}^1M_s$ is reflexive and the dual sequence $\xi_s^\vee$ is short exact. Since ${}^3M_s$ is reflexive the double dual $\xi_s^{\vee\vee}$ is short exact too (consider $\xi_s \to \xi_s^{\vee\vee}$). In particular $\operatorname{Ext}^1_A({}^1M_s^\vee, A) = 0$ as $\operatorname{Ext}^1_A({}^2M_s^\vee, A) = 0$. The rest of the statement now follows from the long exact sequences. (i) is left to the reader. $\square$

4. n-stably reflexive complexes

We define an n-stably reflexive complex with respect to a flat ring homomorphism $S \to R$. In the coherent case with $n > 1$ it is shown that this notion induces an n-stably reflexive module M and conversely, given M, that such a complex exists. We also show that M is an $(n{+}1)$-syzygy of an S-flat finite R-module with a one-sided cohomology condition. This too has a converse.

Definition 4.1. Let n be a positive integer and A a ring.

(i) Let $E\colon \ldots \to E^{-1} \xrightarrow{d^{-1}} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \ldots$ be a complex of finite projective A-modules. Let $E^\vee$ denote the *dual complex*, i.e. $(E^\vee)^i = (E^{1-i})^\vee$ for all i. If $M \cong \operatorname{coker} d^{-1}$ then E is a *hull* for the A-module M. If $\mathrm{H}^i(E) = 0 = \mathrm{H}^i(E^\vee)$ for all $i \leqslant n$ then E is an *n-stably reflexive* A-complex. If E is n-stably reflexive for all n then E is *stably reflexive.*
(ii) Let $h\colon S \to R$ be a flat ring homomorphism. A complex E of finite projective R-modules is *n-stably reflexive with respect to h* if the base change $E_s = E{\otimes_S}k(s)$ is an n-stably reflexive $R_s = R{\otimes_S}k(s)$-complex for all $s \in \operatorname{im}\{\mathfrak{m}\text{-Spec}\, R \to \operatorname{Spec} S\}$. The complex E is *stably reflexive with respect to h* if it's n-stably reflexive with respect to h for all n.

Proposition 4.2. *Let $h\colon S \to R$ be a flat ring homomorphism and M a coherent R-module. Suppose R is coherent, S is noetherian and $n > 1$.*

(i) *There exists a complex E of finite projective R-modules which is a hull for M such that*: *If M is n-stably reflexive with respect to h then so is E.*
(ii) *Let E be an n-stably reflexive complex with respect to h and a hull for M.*
   (a) *The R-module M is n-stably reflexive with respect to h.*

(b) *For any ring homomorphism $S \to S'$ the base change $E' = E \otimes_S S'$ is n-stably reflexive with respect to $h' = h \otimes_S S'$ and as $R' = R \otimes_S S'$-complex, and a hull for $M' = M \otimes_S S'$.*

Proof. (ii) Let $s \in \operatorname{im}\{\mathfrak{m}\text{-Spec}\, R \to \operatorname{Spec} S\}$. Note that $(E^\vee)_s \cong (E_s)^\vee$ since E consists of finite projective modules. By applying Proposition 2.2 to parts of E and $E^\vee$, see Example 2.3, (b) follows. Moreover M is S-flat since $\mathrm{H}^0(E_s) = 0 = \mathrm{H}^1(E_s)$. In addition $\mathrm{H}^{-1}(E_s) = 0$ so $(E_s^2)^\vee \to (E_s^1)^\vee \to M_s^\vee \to 0$ is an exact sequence. Hence its dual $0 \to M_s^{\vee\vee} \to E_s^1 \to E_s^2$ is exact and M_s is reflexive. Moreover $\operatorname{Ext}^i_{R_s}(M_s, R_s) \cong \mathrm{H}^{i+1}(E_s^\vee)$ and $\operatorname{Ext}^i_{R_s}(M_s^\vee, R_s) \cong \mathrm{H}^{i+1}(E_s)$ for $i > 0$ and so M is n-stably reflexive with respect to h.

(i) Since R and M are coherent, so is $M^\vee$. Pic resolutions $P \twoheadrightarrow M$ and $Q \twoheadrightarrow M^\vee$ by finite projective modules. Splicing P with $Q^\vee$ along $P^0 \twoheadrightarrow M \to M^{\vee\vee} \hookrightarrow (Q^0)^\vee$ gives E. Since M is S-flat $\mathrm{H}^i(E_s) \cong \operatorname{Tor}^S_{-i}(M, k(s))$ is 0 for all $i < 0$. Then $\mathrm{H}^i(E_s^\vee) \cong \operatorname{Ext}^{i-1}_{R_s}(M_s, R_s)$ for all $i > 1$, hence $\mathrm{H}^i(E_s^\vee) = 0$ for $1 < i \leqslant n$ and $\ker d^1_{E_s^\vee} \cong M_s^\vee$. By Proposition 3.5 $M^\vee$ is n-stably reflexive over S too. In particular $M^\vee$ is S-flat, therefore $\mathrm{H}^i(E_s^\vee) = 0$ for $i < 0$ and $\operatorname{coker} d^{-1}_{E_s^\vee} \cong M_s^\vee$. Combined $\mathrm{H}^i(E_s^\vee) = 0$ for all $i \leqslant n$. By the symmetry $\mathrm{H}^i(E_s) = 0$ for all $i \leqslant n$. □

Definition 4.3. Let A be a ring. An A-module N is *left n-orthogonal to A* if $\operatorname{Ext}^i_A(N, A) = 0$ for all $0 < i \leqslant n$ and is *left orthogonal to A* if it's left n-orthogonal to A for all n.

Let $h\colon S \to R$ be a flat ring homomorphism and N an R-module. Then N is *left n-orthogonal to h* if N is S-flat and N_s is left n-orthogonal to R_s for all $s \in \operatorname{im}\{\mathfrak{m}\text{-Spec}\, R \to \operatorname{Spec} S\}$. If N is left n-orthogonal to h for all n then N is *left orthogonal to h*.

Proposition 4.4. *Let $h\colon S \to R$ be a flat ring homomorphism and M a coherent R-module. Assume R is coherent, S is noetherian and n is a positive integer. Let E be an R-complex which is a hull for M as given in* Proposition 4.2 (i).

(i) *If N is a coherent R-module which is left $2n$-orthogonal to h then* $\operatorname{Syz}^R_{n+1} N$ *is n-stably reflexive with respect to h.*

(ii) *Assume $n > 1$. The module M is n-stably reflexive with respect to h if and only if $N := \operatorname{coker}\{E^n \to E^{n+1}\}$ is left $2n$-orthogonal to h.*

(iii) *The module M is stably reflexive with respect to h if and only if $N^j :=$* $\operatorname{coker}\{E^{j-1} \to E^j\}$ *is left orthogonal to h for all $j \in \mathbb{Z}$. In this case N^j is stably reflexive with respect to h for all j.*

Proof. (i) Let $\dots \xrightarrow{d^{-2}} F^{-1} \xrightarrow{d^{-1}} F^0 \to N$ be a resolution of N by finite projective modules. Put $M = \operatorname{coker} d^{-(n+2)}$ and let $s \in \operatorname{im}\{\mathfrak{m}\text{-Spec}\, R \to \operatorname{Spec} S\}$. Then M is S-flat. For $i > 0$ one has $\operatorname{Ext}^i_{R_s}(M_s, R_s) \cong \operatorname{Ext}^{n+i+1}_{R_s}(N_s, R_s)$ which is 0 for $0 < i < n$. Let L be $\operatorname{coker}\{(F^{-2n})^\vee \to (F^{-(2n+1)})^\vee\}$. By assumption the sequence (with $d = d_{F_s^\vee}$)

$$0 \leftarrow L_s \leftarrow (F_s^{-(2n+1)})^\vee \xleftarrow{d^{2n+1}} (F_s^{-2n})^\vee \leftarrow \dots \xleftarrow{d^1} (F_s^0)^\vee \leftarrow N_s^\vee \leftarrow 0 \tag{4.4.1}$$

is exact. By Proposition 2.2, see Example 2.3, L is S-flat and we obtain a projective $(2n+1)$-presentation of L. Moreover:

$$M_s^\vee \cong \ker d^{n+2}_{F_s^\vee} \cong \operatorname{coker} d^n_{F_s^\vee} \cong (M^\vee)_s \tag{4.4.2}$$

This implies that $M^\vee \cong \operatorname{Syz}^R_{n+1} L$ and for $i > 0$ $\operatorname{Ext}^i_{R_s}(M_s^\vee, R_s) \cong \mathrm{H}^{-(n-i)}(F_s)$ which is 0 for $0 < i < n$. This also implies $M_s \cong M_s^{\vee\vee}$.

(ii) One direction is (i). For the other direction note that

$$\dots \longrightarrow E_s^n \longrightarrow E_s^{n+1} \longrightarrow N_s \to 0 \tag{4.4.3}$$

is a R_s-projective resolution of N_s since E is n-stably reflexive over S by Proposition 4.2 (i). Then $\operatorname{Ext}^i_{R_s}(N_s, R_s) \cong \mathrm{H}^i(E_s^\vee[-n])$ which is 0 for $i \leqslant 2n$. Define a cohomological δ-functor by $F^q(I) = \mathrm{H}^q((E^1 \to \ldots \to E^{n+1})\otimes_S I)$. Proposition 2.2 implies that $F^{n+1}(S) = N$ is S-flat.

(iii) By Proposition 4.2, M stably reflexive with respect to h implies that $E_s^\vee$ is acyclic and hence N^j is left orthogonal to h for all j. The reverse implication follows from (ii). As $M = N^0$ the second part is clear by 'translational symmetry'. □

Example 4.5. Let $h\colon S \to R$ be a flat homomorphism of noetherian rings. Let d denote the minimal depth at a maximal ideal of $R_s = R\otimes_S k(s)$ for all $s \in Z = \operatorname{im}\{\mathfrak{m}\text{-}\operatorname{Spec} R \to \operatorname{Spec} S\}$. Suppose N is an S-flat finite R-module with $\dim N_s = 0$ for all $s \in Z$. Then $\operatorname{Syz}^R_{n+1} N$ is n-stably reflexive with respect to h for all $n > 0$ with $2n < d$.

Remark 4.6. The complex E in Knudsen's Theorem 2 (2) in [20, Appendix] (where S and R are noetherian) is a hull of the R-module M in our sense such that $E\otimes_S I$ and $E^\vee\otimes_S I$ are acyclic for all S-modules I. In particular E is stably reflexive with respect to h as in Definition 4.1. Conversely, a stably reflexive complex satisfies Knudsen's conditions by Proposition 4.2 (ii b).

In Knudsen's Theorem 2 (3) the hull E should be acyclic and have S-flat $N^j = \operatorname{coker}\{E^{j-1} \to E^j\}$ and $(N^j)^\vee$, and $\operatorname{Ext}^i_R(N^j, R) = 0 = \operatorname{Ext}^i_R((N^j)^\vee, R)$, for all $j \in \mathbb{Z}$. In particular any base changes of E and $E^\vee$ are acyclic (break the complex into short exact sequences). Since $(E^\vee)_s \cong E_s^\vee$ we get $((N^j)^\vee)_s \cong (N_s^j)^\vee$ and N_s^j is left orthogonal to R_s for all $s \in \operatorname{Spec} S$ and all j. Conversely, if N^j is left orthogonal to h for all j then E satisfies the properties in Theorem 2 (3) in [20, Appendix] by Proposition 4.4 (iii).

In view of Remark 3.6, we conclude that Knudsen's Theorem 2 in [20, Appendix] is generalised in Proposition 3.5, 4.2 and 4.4.

Definition 4.7. Suppose T is a noetherian ring and f is a T-regular element. *A matrix factorisation of* f is a pair (φ, ψ) of T-linear maps $\varphi\colon G \to F$ and $\psi\colon F \to G$ between finite T-free modules such that $\varphi\psi = f\cdot \mathrm{id}_F$ and $\varphi\psi = f\cdot \mathrm{id}_G$.

It follows that φ and ψ are injective and that $\operatorname{rk} F = \operatorname{rk} G$. It is also sufficient to check one of the equations.

Corollary 4.8. *Let T be a noetherian ring and suppose (φ, ψ) is a matrix factorisation of a T-regular element f. Put $R = T/(f)$. Reduction by $-\otimes_T R$ of (φ, ψ) gives an acyclic 2-periodic complex of free R-modules*

$$C(\varphi, \psi)\colon \qquad \ldots \xleftarrow{\bar\psi} \bar F \xleftarrow{\bar\varphi} \bar G \xleftarrow{\bar\psi} \bar F \xleftarrow{\bar\varphi} \bar G \xleftarrow{\bar\psi} \ldots .$$

In addition, suppose $S \to T$ is a flat ring homomorphism such that the image f_s of f is a T_s-regular element for all $s \in Z = \operatorname{im}\{\mathfrak{m}\text{-}\operatorname{Spec} T \to \operatorname{Spec} S\}$. Then the induced ring homomorphism $h\colon S \to T/(f)$ is flat. Moreover; $C(\varphi, \psi)$ is a stably reflexive complex with respect to h and a hull for the R-module $\operatorname{coker}\varphi$ *which is stably reflexive with respect to h.*

Proof. The first part is [9, 5.1]. Base change (φ_s, ψ_s) of (φ, ψ) to T_s is a matrix factorisation of f_s. By Proposition 2.2, see Example 2.3, R is S-flat. Since the dual of the complex $C(\varphi, \psi)$ is the complex $C(\varphi^\vee, \psi^\vee)$, we conclude from the first part of the statement and Proposition 4.2. □

5. Approximation

We prove approximation theorems with n-stably reflexive modules resembling Cohen-Macaulay approximation in a flat family.

Let $\mathsf{Alg}^{\mathrm{fl}}$ be the category with objects faithfully flat finite type algebras $h\colon S \to R$ of noetherian rings and with arrows $(g, f)\colon h_1 \to h_2$ pairs of ring homomorphisms $g\colon S_1 \to S_2$ and $f\colon R_1 \to R_2$ such that $h_2 g = f h_1$ and such that the induced map $f\otimes 1\colon R_1\otimes S_2 \to R_2$ is an isomorphism:

$$\begin{tikzcd} R_1 \arrow[r, "f"] & R_2 & R_1\otimes_{S_1} S_2 \arrow[l, "\simeq"'] \\ S_1 \arrow[u, "h_1"] \arrow[r, "g"] & S_2 \arrow[u, "h_2"'] & \end{tikzcd}$$

Let NR denote the category of noetherian rings. The forgetful functor $p\colon \mathsf{Alg}^{\mathrm{fl}} \to \mathsf{NR}$; $(g, f) \mapsto g$, makes $\mathsf{Alg}^{\mathrm{fl}}$ a category fibred in groupoids over NR.[1]

Let $\mathsf{mod}^{\mathrm{fl}}$ be the category of pairs $(h\colon S \to R, N)$ with h in $\mathsf{Alg}^{\mathrm{fl}}$ and N an S-flat finite R-module. A morphism $(h_1, N_1) \to (h_2, N_2)$ is a morphism $(g, f)\colon h_1 \to h_2$ in $\mathsf{Alg}^{\mathrm{fl}}$ and a f-linear map $\alpha\colon N_1 \to N_2$. Then α is cocartesian with respect to the forgetful functor $F\colon \mathsf{mod}^{\mathrm{fl}} \to \mathsf{Alg}^{\mathrm{fl}}$ if $\alpha\otimes 1\colon N_1\otimes S_2 \to N_2$ is an isomorphism. All objects admit arbitrary base change. It follows that $\mathsf{mod}^{\mathrm{fl}}$ is a fibred category and it is *fibred in additive categories* over $\mathsf{Alg}^{\mathrm{fl}}$ since the fibre categories (denoted $\mathsf{mod}^{\mathrm{fl}}_h$) are additive and the fibre of homomorphisms are additive groups with bilinear composition (cf. [18, 3.2] for a precise definition).

Given integers $n \geqslant 0$, $r, s > 0$. We define some full subcategories of $\mathsf{mod}^{\mathrm{fl}}$ by properties of their objects $(h\colon S \to R, N)$ as follows.

Category	*Property*
P	N is R-projective
$\hat{\mathsf{P}}^{\mathrm{fl}}(n)$	N has an R-projective resolution of length $\leqslant n$
$\mathsf{X}(r)$	N is r-stably reflexive with respect to h
$\mathsf{Y}(r; s)$	N is isomorphic to a direct sum of R-modules $\oplus N_i$ such that for all i $\mathrm{Syz}^R_{n_i} N_i$ is $(r + n_i)$-stably reflexive w.r.t. h for some $0 \leqslant n_i < s$

The definitions are also meaningful in the case h is a flat homomorphism of local noetherian rings (and N is S-flat and R-finite). By Schanuel's lemma the choice of syzygy is irrelevant for the definition of $\mathsf{Y}(r; s)$. If $\mathrm{Syz}^R_n N$ is $(r + n)$-stably reflexive over S and $r + n > 1$ then $\mathrm{Syz}^R_{n+1} N$ is $(r + n - 1)$-stably reflexive over S by Lemma 3.8. Flatness combined with Proposition 3.5 implies that all the categories are fibered in additive categories over $\mathsf{Alg}^{\mathrm{fl}}$ for $r > 1$. They all contain P as a subcategory fibred in additive categories and there are corresponding quotient categories $\mathsf{mod}^{\mathrm{fl}}/\mathsf{P}, \dots$ which all are fibred in additive categories over $\mathsf{Alg}^{\mathrm{fl}}$ by [18, 3.4].

Let X be a subcategory of a category Y. An arrow $\pi\colon M \to N$ in Y is called a *right* X*-approximation of* N if M is in X and any $M' \to N$ with M' in X factorises through π. Dually, $\iota\colon N \to L$ is called a *left* X*-approximation of* N if L is in X and any $N \to L'$ with L' in X factorises through ι. The approximations need not be unique. The subcategory X is *contravariantly* (*covariantly*) *finite in* Y if every object N in Y has a right (left) X-approximation.

An arrow $\pi\colon M \to N$ in Y is called *right minimal* if for any $\eta\colon M \to M$ with $\pi\eta = \pi$ it follows that η is an automorphism. Dually, π is called *left minimal* if for any $\theta\colon N \to N$ with $\theta\pi = \pi$ it follows that θ is an automorphism. Note that if $\pi\colon M \to N$ and $\pi'\colon M' \to N$ both are right minimal then there exists an isomorphism $\varphi\colon M \to M'$ with $\pi = \pi'\varphi$, and similarly for left minimal morphisms.

[1] I.e. $p^{\mathrm{op}}\colon (\mathsf{Alg}^{\mathrm{fl}})^{\mathrm{op}} \to \mathsf{NR}^{\mathrm{op}}$ is a fibred category as defined by A. Vistoli in [11]. Some would call p *cofibred* in groupoids.

We will simply call a right (left) X-approximation for minimal if it is right (left) minimal.

In the unpublished manuscript [8] R.-O. Buchweitz proved an approximation theorem for (not necessarily commutative) Gorenstein rings by applying a resolution of a complex. This is also the basic construction in the proof of the following result.

Theorem 5.1. *Given an $h\colon S \to R$ in $\mathsf{Alg}^{\mathrm{fl}}$ and suppose $r, s > 1$.*

(i) *For any module N in $\mathsf{Y}(r;s)_h$ there are short exact sequences*
  (a) $0 \to L \longrightarrow M \longrightarrow N \to 0$ *with L in $\hat{\mathsf{P}}^{\mathrm{fl}}(s-2)_h$ and M in $\mathsf{X}(r)_h$*
  (b) $0 \to N \longrightarrow L' \longrightarrow M' \to 0$ *with L' in $\hat{\mathsf{P}}^{\mathrm{fl}}(s-1)_h$ and M' in $\mathsf{X}(r-1)_h$*
  which are preserved by any base change in NR.
(ii) *If $s \leqslant r$ then* (a) *is a right $\mathsf{X}(r)_h$-approximation of N and if $s \leqslant r-2$ then* (b) *is a left $\hat{\mathsf{P}}^{\mathrm{fl}}(s-1)_h$-approximation of N.*
(iii) *In particular $\mathsf{X}(r)_h$ is contravariantly finite in $\mathsf{Y}(r,r)_h$ and $\hat{\mathsf{P}}^{\mathrm{fl}}(r-3)_h$ is covariantly finite in $\mathsf{Y}(r, r-2)_h$.*
(iv) *If h is a flat homomorphism of local noetherian rings then there exists minimal approximations* (a) *and* (b).

Proof. A direct sum of short exact sequences as in (a) (or in (b)) gives a new short exact sequence of the same kind. All statements in Theorem 5.1 are independent of the n_i appearing in the definition of $\mathsf{Y}(r,s)$. We therefore assume that $\mathrm{Syz}^R_n N$ is m-stably reflexive with respect to h where $m = r+n$. We also assume that $n > 0$ and leave the case $n = 0$ to the reader. Let $P \twoheadrightarrow N$ be an R-projective resolution of N. Recall our convention $(P^\vee)^{n+1} = (P^{-n})^\vee$. By Proposition 3.5 the dual complex $P^\vee$ is acyclic in degrees between $n+2$ and $n+m$. Choose a projective resolution $Q \to \tau^{\leqslant n+1} P^\vee$ of the soft truncation. Let C denote the mapping cone $C(f)$ of the corresponding map $f\colon Q \to P^\vee$. In particular $\tau^{\leqslant n+1} C$ is a projective resolution of $(\mathrm{Syz}^R_n N)^\vee$ and by Proposition 3.5 $C^\vee$ is acyclic in degrees between $-(n-2)$ and r. In addition $\mathrm{Syz}^R_n N$ is reflexive, and it follows that $C^\vee$ and $Q^\vee$ are acyclic in negative degrees and $\mathrm{H}^0(Q^\vee) \cong N$. We obtain the following defining diagram

(5.1.1)
$$\begin{tikzcd}
\dots \arrow[r, "d^{-2}"] & (Q^2)^\vee \arrow[rr, "d^{-1}"] \arrow[d] & & (Q^1)^\vee \arrow[rr, "d^0"] \arrow[dr] & & (Q^0)^\vee \arrow[r, "d^1"] & \dots \\
& L \arrow[r] & M \arrow[ur] \arrow[dr] & & L' \arrow[r] & M' \arrow[u] & \\
& & & N \arrow[ur] & & &
\end{tikzcd}$$

with $L := \operatorname{coker} d^{-2}$, $M := \ker d^0$, $L' := \operatorname{coker} d^{-1}$ and $M' := \ker d^1$ and where $\operatorname{pdim} L \leqslant n-1$ and $\operatorname{pdim} L' \leqslant n$. Since $Q[1]^\vee$ equals $C^\vee$ in degrees $\geqslant 1$ we have that M is the $(r+1)$-syzygy of $K := \operatorname{coker}\{(C^{-r+1})^\vee \to (C^{-r})^\vee\}$. Since N is S-flat so is $\mathrm{Syz}^R_n N$ and $(\mathrm{Syz}^R_n N)^\vee$. Let $S \to S'$ be a ring homomorphism and let $N' = N \otimes S'$ denote the induced $R' = R \otimes S$-module. By Proposition 3.5 $\mathrm{Syz}^R_n N' \cong (\mathrm{Syz}^R_n N) \otimes S'$ is m-stably reflexive and likewise for $(\mathrm{Syz}^R_n N')^\vee$. It follows that the base change $C' = C \otimes S'$ and $(C')^\vee \cong (C^\vee)'$ retains the properties of C and $C^\vee$. In particular we get that a truncation of $(C')^\vee$ gives an R'-projective resolution of $K' = K \otimes S'$ and K is S-flat. The estimates above give $\mathrm{Ext}^i_{R'}(K', R') = \mathrm{H}^{i-r}(C') = 0$ for $0 < i < 2(r+n)+1$. Hence K is $2r$-orthogonal to h. By Proposition 4.4 M is r-stably reflexive with respect to h. This argument also gives that $M' \cong \mathrm{Syz}^R_r K$ is $(r-1)$-stably reflexive with respect to h. We note that L' and L are S-flat with finite projective resolutions given by the truncations of $Q^\vee$ of length n and $n-1$ respectively.

Let L_1 and M_1 be R-modules with L_1 of finite projective dimension and M_1 r-stably reflexive. By induction on $\operatorname{pdim} L_1$ we get

$$\operatorname{Ext}^i_R(M_1, L_1) = 0 \text{ for } 0 < i < r - \operatorname{pdim} L_1. \tag{5.1.2}$$

Any R-linear map $M_1 \to N$ lifts by (5.1.2) to a map $M_1 \to M$ in (a) if $r \geqslant s$ since this implies $r - \operatorname{pdim} L > 1$. If $\operatorname{pdim} L_1 \leqslant s - 1$ then any map $N \to L_1$ extends by (5.1.2) to a map $L' \to L_1$ in (b) if $r \geqslant s+2$ since this implies $(r-1) - \operatorname{pdim} L_1 > 1$.

In the local case we choose Q to be a minimal complex of free modules. Then (a) and (b) are minimal approximations by a result analogous to [18, 6.2]. □

A morphism $\mathsf{A}_1 \to \mathsf{A}_2 \to \mathsf{A}_3$ of categories fibred in additive categories over some base category is a *short exact sequence* if $\mathsf{A}_1 \to \mathsf{A}_2$ is an inclusion and $\mathsf{A}_2 \to \mathsf{A}_3$ is equivalent to the quotient morphism $\mathsf{A}_2 \to \mathsf{A}_2/\mathsf{A}_1$.

Theorem 5.2. *Suppose $r > 2$ and $s > 1$. Then:*

(i) *For any $s \leqslant r$ the $\mathsf{X}(r)$-approximation induces a morphism of categories fibred in additive categories $j^! \colon \mathsf{Y}(r;s)/\mathsf{P} \to \mathsf{X}(r)/\mathsf{P}$ which is a right adjoint to the full and faithful inclusion morphism $j_! \colon \mathsf{X}(r)/\mathsf{P} \to \mathsf{Y}(r;s)/\mathsf{P}$.*

(ii) *For any $s \leqslant r-2$ the $\hat{\mathsf{P}}^{\mathrm{fl}}(s-1)$-approximation induces a morphism of categories fibred in additive categories $i^* \colon \mathsf{Y}(r;s)/\mathsf{P} \to \hat{\mathsf{P}}^{\mathrm{fl}}(s-1)/\mathsf{P}$ which is a left adjoint to the full and faithful inclusion morphism $i_* \colon \hat{\mathsf{P}}^{\mathrm{fl}}(s-1)/\mathsf{P} \to \mathsf{Y}(r;s)/\mathsf{P}$.*

(iii) *Together these maps give the following commutative diagram of short exact sequences of categories fibred in additive categories:*

$$\begin{tikzcd}
0 \arrow[r] & \hat{\mathsf{P}}^{\mathrm{fl}}(r-3)/\mathsf{P} \arrow[r, "i_*"] \arrow[d, "\mathrm{id}"'] & \mathsf{Y}(r;r-2)/\mathsf{P} \arrow[r, "j^!"] \arrow[d, equal] & \mathsf{X}(r)/\mathsf{P} \arrow[r] & 0 \\
0 & \hat{\mathsf{P}}^{\mathrm{fl}}(r-3)/\mathsf{P} \arrow[l] & \mathsf{Y}(r;r-2)/\mathsf{P} \arrow[l, "i^*"'] & \mathsf{X}(r)/\mathsf{P} \arrow[l, "j_!"'] \arrow[u, "\mathrm{id}"'] & 0 \arrow[l]
\end{tikzcd}$$

Proof. The argument very much resembles the proof of [18, 4.5], but where the Auslander-Buchweitz axioms are substituted by the appropriate Ext-vanishing which again is based on Proposition 3.5. By base change all arguments can be reduced to statements about the fibre categories. This is done as in [18, 4.5]. In the fibre categories there are many similar arguments. E.g. to define the functor $j^!$ we choose right $\mathsf{X}(r)$-approximations $\pi_i \colon M_i \to N_i$ as in (a) in Theorem 5.1 for all modules N_i in $\mathsf{Y}(r;s)$ with $\pi_i = \mathrm{id}$ if N_i is contained in $\mathsf{X}(r)$. For each map $\varphi_{21} \colon N_1 \to N_2$ choose a lifting $\psi_{21} \colon M_1 \to M_2$ of φ_{21} (assured by Theorem 5.1 ii) with $\psi_{21} = \varphi_{21}$ in the case $N_i = M_i$. Then $j^!$ is defined by $[\varphi_{21}] \mapsto [\psi_{21}]$. One has to prove that composition is well defined: After reducing to a fibre category, $\psi_{32}\psi_{21} - \psi_{31}$ factors through a $\delta \colon M_1 \to L_3$ and L_3 sits, by Theorem 5.1, in a short exact sequence $L'' \to P \to L_3$ with P projective and $\operatorname{pdim} L'' \leqslant s-3$. Hence $\operatorname{Ext}^1(M_1, L'') = 0$ by (5.1.2) since $r - (s-3) > 1$ and δ factorises through P and $[\psi_{32}\psi_{21}] = [\psi_{31}]$. In a similar way one shows that $j^!$ is well defined in terms of representatives. The definition and arguments in the i^*-case are analogous.

Let $\pi \colon M \to N$ be a chosen $\mathsf{X}(r)_h$-approximation. The adjointness map

$$\pi_* \colon \operatorname{Hom}(M_1, j^! N) \longrightarrow \operatorname{Hom}(j_! M_1, N) \tag{5.2.1}$$

is surjective for any M_1 in $\mathsf{X}(r)$ since $\operatorname{Ext}^1(M_1, L) = 0$ by (5.1.2) as $r - (s-1) > 1$. Injectivity is similar to composition and reduces to the same estimate. The other adjoint pair is analogous.

Exactness in the lower row of the diagram: After base change to a fibre category let $\lambda \colon L_1' \to L_2'$ be the chosen extension of $\varphi \colon N_1 \to N_2$. By definition $i^*[\varphi] = [\lambda]$.

Assume $[\lambda] = 0$, i.e. λ factors through a projective module. Naturality of long-exact sequences gives a commutative diagram:

$$\begin{tikzcd} \mathrm{Ext}^1(L_2', L_2) \arrow[r, "\lambda^*"] \arrow[d] & \mathrm{Ext}^1(L_1', L_2) \arrow[d] \\ \mathrm{Ext}^1(N_2, L_2) \arrow[r, "\varphi^*"] & \mathrm{Ext}^1(N_1, L_2) \end{tikzcd} \tag{5.2.2}$$

We have $\mathrm{Ext}^i(M_j', L_2) = 0$ for $i = 1, 2$ by (5.1.2) and the vertical maps are isomorphisms. Hence $\lambda^* = 0 \Rightarrow \varphi^* = 0$. Let e_2 denote the short exact sequence $L_2 \to M_2 \to N_2$. Then the short exact sequence $\varphi^* e_2$ splits and hence $\varphi\colon N_1 \to N_2$ factors through M_2 in $\mathsf{X}(r)$ which is what we wanted to show. The other parts are similar (see the proof of [18, 4.5] for hints). □

Finally we give the relative Cohen-Macaulay approximation result which is the 'limit' of Theorems 5.1 and 5.2. Define full subcategories of $\mathsf{mod}^{\mathrm{fl}}$ by objects $(h\colon S \to R, N)$ as follows.

Category	*Property*
$\hat{\mathsf{P}}^{\mathrm{fl}}$	N has a finite R-projective resolution
X	N is stably reflexive with respect to h
Y	$(h, \mathrm{Syz}_n^R N)$ is in X for an integer $n \geqslant 0$ depending on N

We note that X, Y and $\hat{\mathsf{P}}^{\mathrm{fl}}$ are fibred in additive categories over $\mathsf{Alg}^{\mathrm{fl}}$. The expression 'without the parametres' in the following theorem is short for 'after removing the parametres from the symbols in the statements (e.g. $\mathsf{Y}(r,s)_h$ becomes Y_h) and removing the conditions involving parametres'.

Theorem 5.3. *The statements in* Theorem 5.1 *and* Theorem 5.2 *without the parametres are true.*

Proof. One checks that all four of the Auslander-Buchweitz axioms (cf. [18]) hold in the fibre categories. The main ingredient here is Proposition 3.5. One also checks that the two base change axioms in [18] hold (formally one also has to consider the category mod fibred in abelian categories over $\mathsf{Alg}^{\mathrm{fl}}$ of pairs (h, N) where N is not necessarily S-flat). Then this is a corollary of [18, 4.4 and 4.5]. Alternatively one can follow the proofs of Theorems 5.1 and 5.2. □

If C is an additive subcategory of a module category then $\hat{\mathsf{C}}$ denotes the full subcategory of modules N which have a finite C-resolution $0 \to C^{-n} \to \dots \to C^0 \to N \to 0$. The minimal n is the C-resolving dimension $\mathsf{C}\text{-}\operatorname{res.dim} N$ of N. Let A be a coherent ring and denote by X_A the category of stably reflexive A-modules and by Y_A the category of coherent A-modules N with $\mathrm{Syz}_n^A N$ in X_A for some n. If A is noetherian and N a finite A-module, $\mathsf{X}_A\text{-}\operatorname{res.dim} N$ equals the *Gorenstein dimension* of N defined by Auslander and Bridger and $\mathsf{Y}_A = \hat{\mathsf{X}}_A$; see [4, 3.13]. This extends to the relative setting and gives a characterisation of Y.

Lemma 5.4. *Let $h\colon S \to R$ be a ring homomorphism in $\mathsf{Alg}^{\mathrm{fl}}$ and N a module in $\mathsf{mod}_h^{\mathrm{fl}}$. Put $Z = \operatorname{im}\{\mathfrak{m}\text{-}\operatorname{Spec} R \to \operatorname{Spec} S\}$. The following are equivalent:*

(i) $\mathrm{Syz}_n^R N$ *is in* X_h.
(i') $\mathrm{Syz}_n^{R_s} N_s$ *is in* X_{R_s} *for all* $s \in Z$.
(ii) $\mathsf{X}_h\text{-}\operatorname{res.dim} N \leqslant n$.
(ii') $\mathsf{X}_{R_s}\text{-}\operatorname{res.dim} N_s \leqslant n$ *for all* $s \in Z$.

Proof. Put $M = \mathrm{Syz}_n^R N$. Since N is S-flat M is S-flat too and M_s is (stably) isomorphic to $\mathrm{Syz}_n^{R_s} N_s$. Hence (i')⇔(i). Since $\mathsf{P}_h \subseteq \mathsf{X}_h$ (i)⇒(ii). The fibre at s

of an X_h-resolution of N gives an X_{R_s}-resolution of N_s hence (ii)$\Rightarrow$(ii'). Finally (ii')$\Leftrightarrow$(i') by [4, 3.13]. $\square$

To shed some further light on Y we give the following perhaps not so well known results of Auslander and Bridger (who attribute (i) to C. Peskine and L. Szpiro).

Proposition 5.5 ([4, 4.12, 13, 35])**.** *Let A be a noetherian local ring and N a finite A-module.*

(i) *Suppose (f) is an A-regular sequence of length n which annihilates N. Then $\mathsf{X}_A\text{-}\operatorname{res.dim} N = \operatorname{grade}_A N = n$ if and only if N is in $\mathsf{X}_{A/(f)}$.*
(ii) *If N is in $\hat{\mathsf{X}}_A$ then $\mathsf{X}_A\text{-}\operatorname{res.dim} N + \operatorname{depth} N = \operatorname{depth} A$.*
(iii) *If N is in $\hat{\mathsf{X}}_A$ then $\mathsf{X}_A\text{-}\operatorname{res.dim} N = \min\{n \mid \operatorname{Ext}^i_A(N, A) = 0$ for all $i > n\}$.*

These results have recently been generalised to coherent rings; see [17].

6. Pointed Gorenstein singularities and Knudsen's lemma

We state and prove our version of Knudsen's lemma. We also give a general result about versal families for deformations of pointed algebras and make it explicit for isolated complete intersection singularities. Finally we generalise Knudsen's stabilisation to pointed plane curve singularities.

Theorem 6.1. *Let $h\colon S \to R$ be a flat homomorphism of local, noetherian rings and let k denote the residue field $S/\mathfrak{m}_S$ and A the central fibre $R{\otimes}_S k$. Given an S-algebra map $R \to S$, let I denote the kernel and $I^\vee = \operatorname{Hom}_R(I, R)$. Assume A is Gorenstein of dimension 1. Then:*

(i) *The R-module I is stably reflexive with respect to h.*
(ii) *The R-module $I^\vee/R$ is isomorphic to S.*
(iii) *The R-submodule J of the total quotient ring $K(R)$ consisting of elements that multiply I into R is isomorphic to $I^\vee$.*
(iv) *The image of the pairing $I{\otimes}_R I^\vee \to R$ equals R if A is a regular ring and I if not. In particular $I^\vee{\otimes}_R S$ is a free S-module which has rank 1 in the regular case and rank 2 otherwise.*

Proof. (i) We have $I{\otimes}_S k \cong \mathfrak{m}_A$ and I is S-flat. Since $\mathfrak{m}_A$ is a maximal Cohen-Macaulay A-module and A is Gorenstein, I is stably reflexive with respect to h by Corollary 3.7.

(ii) Applying $\operatorname{Hom}_R(-, R)$ to $I \to R \to S$ gives the short exact sequence $R \to I^\vee \to \operatorname{Ext}^1_R(S, R)$. Since A is 1-dimensional Gorenstein we have

$$\operatorname{Hom}_A(\mathfrak{m}_A, A)/A \cong \operatorname{Ext}^1_A(A/\mathfrak{m}_A, A) \cong A/\mathfrak{m}_A \cong k \tag{6.1.1}$$

and $\operatorname{Ext}^i_A(k, A) = 0$ for all $i \neq 1$. By Corollary 2.5 we have that $I^\vee/R \cong \operatorname{Ext}^1_R(S, R)$ is an S-flat R-deformation of the A-module k to S and hence is isomorphic to S as one can show as follows. Put $N := I^\vee/R$ and $N_n := N{\otimes}_S S/\mathfrak{m}_S^{n+1}$. By induction we can construct a tower of isomorphisms $\theta_n : S_n \cong N_n$ of R_n-modules starting with $S_0 \cong k \cong N_0$. Pick a lifting in N_n of $\theta_{n-1}(1) \in N_{n-1}$. The resulting map of S_n-modules $S_n \to N_n$ is R_n-linear since N_n is an S_n-module as R_n-module. The S_n-flatness of N_n implies that θ_n is an isomorphism. We get an isomorphism in the limit and by baby Artin approximation (e.g. [18, 6.1]) there is an isomorphism $S \cong N$ of R-modules which agrees with θ_1.

(iii) Consider the map $m\colon J \to I^\vee$ defined by the multiplication map $m_\gamma(u) = \gamma{\cdot}u$ for $\gamma \in J$ and $u \in I$. First we show that m is injective. Suppose there is a $\gamma \neq 0$ such that $\gamma{\cdot}I = 0$. We have $\gamma = ab^{-1}$ for some $a, b \in R$ and I is annihilated by a. The base change map $\operatorname{Hom}_R(R/I, R){\otimes}_S k \to \operatorname{Hom}_A(k, A) = 0$ is an isomorphism by Proposition 2.2. This is a contradiction and m is injective.

Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] \arrow[d, equal] & J \arrow[r] \arrow[d, "m"] & J/R \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & R \arrow[r] & I^\vee \arrow[r] & I^\vee/R \arrow[r] & 0 \end{tikzcd} \tag{6.1.2}$$

To show surjectivity of m we produce an element $\tilde{\varepsilon}$ in J which maps to a generator for $I^\vee/R \cong S$. Consider diagram (6.1.2) for $R = A$. There is a non-zero divisor $x \in \mathfrak{m}_A$. The 0-dimensional quotient ring $A/(x)$ is Gorenstein, i.e. the socle has length 1 generated by an $\bar{f}$ induced by some $f \in \mathfrak{m}_A$. Put $\varepsilon = fx^{-1}$. Since $\varepsilon \notin A$, we have that m_ε maps to a generator for $\operatorname{Hom}_A(\mathfrak{m}_A, A)/A$. Since $I^\vee \to \operatorname{Hom}_A(\mathfrak{m}_A, A)$ is surjective, m_ε lifts to an R-linear map $\psi\colon I \to R$. Pick a lifting $\tilde{x}$ in I of x. It is not a zero divisor (see Example 2.3). Let $\tilde{f} = \psi(\tilde{x})$ and put $\tilde{\varepsilon} = \tilde{f}\tilde{x}^{-1}$. Then $m_{\tilde{\varepsilon}} \in \operatorname{Hom}_R(I, K(R))$. To show that $m_{\tilde{\varepsilon}}$ equals ψ let φ denote the difference $\psi - m_{\tilde{\varepsilon}} \in \operatorname{Hom}_R(I, K(R))$. If $u \in I$, then $\tilde{x}{\cdot}\varphi(u) = \varphi(\tilde{x}u) = u{\cdot}\varphi(\tilde{x}) = 0$. But $\tilde{x}$ is a unit in $K(R)$, hence $\varphi = 0$.

(iv) Let $\mu\colon I{\otimes_R}I^\vee \to R$ denote the pairing. We have have already shown that $\operatorname{im}\mu = I + \varepsilon{\cdot}I \subseteq R$. Assume first $R = A$. If A is regular then $\mathfrak{m}_A$ is a principal ideal and $\operatorname{im}\mu = A$. For the converse suppose $\operatorname{im}\mu = A$, i.e. there is an element $u \in \mathfrak{m}_A$ with $\varepsilon u = 1$. In the proof of (iii) we only assumed that x in $\varepsilon = fx^{-1}$ wasn't a zero divisor. But we can also assume that x cannot be equal to $x_1 y$ for any x_1 and y in $\mathfrak{m}_A$ (e.g. if $x = x_1 y$ then $(x) \subsetneq (x_1)$, repeat with x_1, and so on). Since $fu = x$ we have $f \notin \mathfrak{m}_A$ and $\operatorname{Soc} A/(x) = (\bar{f}) = A/(x)$, i.e. $A/(x)$ is a field, (x) is the maximal ideal of A, and A is regular. Hence if A isn't regular then $\operatorname{im}\mu = \mathfrak{m}_A$. For the general case (with A not regular) note that if $-\otimes_S k$ is applied to $I \xrightarrow{m_{\tilde{\varepsilon}}} R \to R/\tilde{\varepsilon}{\cdot}I$, by the construction of $\tilde{\varepsilon}$ we obtain the short exact sequence $\mathfrak{m}_A \xrightarrow{m_\varepsilon} A \to A/\mathfrak{m}_A$ and $\operatorname{Tor}_1^S(R/\tilde{\varepsilon}{\cdot}I, k) = 0$. Hence $R/\tilde{\varepsilon}{\cdot}I \to A/\mathfrak{m}_A$ is an S-flat R-deformation of k to S and (as in the proof of (ii)) isomorphic to $S \to k$. In particular $\tilde{\varepsilon}{\cdot}I = I$. If A is regular then $m_{\tilde{\varepsilon}}$ is an isomorphism.

For the last part note that the pairing μ composed with the inclusion $j : R \to I^\vee$ equals the multiplication map $u{\otimes}\rho \mapsto u{\cdot}\rho$. Hence $j\mu$ is obtained by applying $-\otimes_R I^\vee$ to the inclusion $I \subseteq R$. By the serpent lemma the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d] & I{\otimes_R}I^\vee \arrow[r, "\mathrm{id}\otimes p"] \arrow[d, "j\mu"] & I{\otimes_R}I^\vee/R \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & R \arrow[r, "j"] & I^\vee \arrow[r, "p"] & I^\vee/R \arrow[r] & 0 \end{tikzcd} \tag{6.1.3}$$

gives the 6 last terms

$$0 \to \operatorname{Tor}_1^R(I^\vee, S) \xrightarrow{p_*} \operatorname{Tor}_1^R(I^\vee/R, S) \xrightarrow{\delta} S \to I^\vee/I{\cdot}I^\vee \to I^\vee/R \to 0 \tag{6.1.4}$$

in the long-exact sequence derived from $-\otimes_R S$ applied to $R \to I^\vee \to I^\vee/R$. We get that $\delta = 0$ iff the image of μ equals I and δ is surjective iff $\operatorname{im}\mu = R$. By (ii) the result follows. $\square$

Remark 6.2. The short exact sequence of R-modules $R \to I^\vee \to S$ in Theorem 6.1 is an example of the right X_h-approximation given in Theorem 5.3. The left $\hat{\mathsf{P}}^{\mathrm{fl}}$-approximation is given as follows. Lift generators of $\mathfrak{m}_A$ to I to define a surjective map $F \to I$ from a finite R-free module F. Dualising the short exact sequence $\operatorname{Syz}^R I \to F \to I$ gives a short exact sequence $I^\vee \to F^\vee \to (\operatorname{Syz}^R I)^\vee$ since $\operatorname{Ext}_R^1(I, R) = 0$ by Corollary 2.5. The cokernel of the composition $R \to I^\vee \to F^\vee$ defines L' giving the left $\hat{\mathsf{P}}^{\mathrm{fl}}$-approximation of S. Both approximations are contained in the following commutative diagram with short exact rows and columns

and the (co)cartesian boxed square:

$$(6.2.1)\qquad \begin{tikzcd} R \arrow[r] \arrow[d, equal] & I^\vee \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d] \\ R \arrow[r] & F^\vee \arrow[r] \arrow[d] & L' \arrow[d] \\ & (\mathrm{Syz}^R I)^\vee \arrow[r, equal] & (\mathrm{Syz}^R I)^\vee \end{tikzcd}$$

Corollary 6.3. *With notation and assumptions as in* Theorem 6.1 *except for the last sentence. Assume that A is essentially of finite type over k and that* $\operatorname{Spec} A\otimes_k\bar{k}$ *only has complete intersection singularities of dimension* 1 *where $\bar{k}$ is an algebraic closure of k. Then the conclusion in* Theorem 6.1 *holds.*

Proof. By [15, 19.3.4] A is a complete intersection and in particular a 1-dimensional Gorenstein ring. The result hence follows from Theorem 6.1. $\square$

Remark 6.4. In Knudsen's lemma [20, 2.2] $\operatorname{Spec} A\otimes_k\bar{k}$ is assumed to have only ordinary double points. Knudsen has given a different proof of Corollary 6.3 (i)-(iii) in this case [21]. Note that (iv) is used in the proof of [20, 3.7].

Let Λ be a noetherian local ring with residue field k. Define ${}_\Lambda\mathsf{H}$ to be the category of noetherian, henselian, local Λ-algebras S with residue field k. Let ${}_\Lambda\mathsf{H}/k$ denote the corresponding comma category. Let ${}_\Lambda\mathsf{HFP}$ be the category with objects *algebraic, flat and pointed algebras* $\xi\colon S\to T\to S$ in ${}_\Lambda\mathsf{H}$, i.e. T is the henselisation of a finite type S-algebra, T is S-flat and the composition is the identity. The morphisms $\xi_1\to\xi_0$ are corresponding commutative diagrams

$$(6.4.1)\qquad \begin{tikzcd} S_1 \arrow[r] \arrow[d, "p"] & T_1 \arrow[r] \arrow[d, "q"] & S_1 \arrow[d, "p"] \\ S_0 \arrow[r] & T_0 \arrow[r] & S_0 \end{tikzcd}$$

such that the left square is cocartesian (then the right square is cocartesian too). Note that base change exists for the forgetful functor ${}_\Lambda\mathsf{HFP}\to{}_\Lambda\mathsf{H}$. It is given by the henselisation of the tensor product $R=R_1\otimes_{S_1}S_2$ in the maximal ideal $\mathfrak{m}_{R_1}R+\mathfrak{m}_{S_2}R$, denoted $R_1\tilde{\otimes}_{S_1}S_2$. The map $R_1\tilde{\otimes}_{S_1}S_2\to S_2$ is the natural one.

Fix an object $\xi\colon k\to A\to k$ in ${}_\Lambda\mathsf{HFP}$. Let $\mathsf{Def}_{A\to k}$ denote the corresponding comma category ${}_\Lambda\mathsf{HFP}/\xi$ of flat and pointed algebras above ξ, called *pointed deformations of A*. Similarly there is a category Def_A of (unpointed) deformations of A. Both categories are fibred categories above ${}_\Lambda\mathsf{H}/k$ and there is a map of fibred categories $\mathsf{Def}_{A\to k}\to\mathsf{Def}_A$ by forgetting the pointing. Let $\mathrm{Def}_{A\to k}\to\mathrm{Def}_A$ denote the associated map of functors. We often abuse the notation by hiding the maps to the base object. For the definition of a (formally) versal element in functors like these we refer to M. Artin [3].

Theorem 6.5. *Given an object $k\to A\to k$ in ${}_\Lambda\mathsf{HFP}$ and assume $\iota\colon S\to R$ is an unpointed deformation of A. Let $\mathrm{id}\,\tilde{\otimes}1\colon R\to R^{(2)}$ be the base change of ι by ι and $R^{(2)}\to R$ the multiplication map. Then $\xi_v\colon R\to R^{(2)}\to R$ is a pointed deformation of A.*

If $\iota\colon S\to R$ gives a formally versal (*respectively versal*) *element in* Def_A *then $R\to R^{(2)}\to R$ gives a formally versal* (*respectively versal*) *element in* $\mathrm{Def}_{A\to k}$.

Proof. The residue field of R is k and $(R^{(2)}\to R)\otimes_R k\cong(A\to k)$. Hence ξ_v induces an object in $\mathsf{Def}_{A\to k}$. To prove (formal) versality we consider a map $\xi_1\to\xi_0$ in $\mathsf{Def}_{R\to k}$ as in (6.4.1) with surjective vertical maps. Given a map $\alpha_0\colon R\to S_0$ such that $\xi_v\tilde{\otimes}_R S_0\cong\xi_0$ we show that there is a lifting $\alpha_1\colon R\to S_1$ of α_0 inducing ξ_1.

I.e. we consider the following lifting diagram:

(6.5.1)
$$
\begin{tikzcd}
 & S_1 \arrow[rr, "\iota_1"] \arrow[d, "p"] & & T_1 \arrow[rr, "\pi_1"] \arrow[d, "q"] & & S_1 \arrow[d, "p"] & \\
 & S_0 \arrow[rr, dashed, "\iota_0"] & & T_0 \arrow[rr, "\pi_0"] & & S_0 & \\
 & & R \arrow[ul, "\alpha_0"] \arrow[uul, dashed, "\alpha_1"] \arrow[rr, dashed, "\mathrm{id}\,\tilde{\otimes}1"] & & R^{(2)} \arrow[ul, "\beta_0"] \arrow[uul, dashed, "\beta_1"] \arrow[rr, "\mu"] & & R \arrow[ul, "\alpha_0"] \arrow[uul, dashed, "\alpha_1"] \\
S \arrow[uur, "\theta_0"] \arrow[uuur, dotted, "\theta_1"] \arrow[urr, "\iota"] \arrow[rr, "\iota"] & & R \arrow[uur, "\tau_0"] \arrow[uuur, dotted, "\tau_1"] \arrow[urr, "1\tilde{\otimes}\,\mathrm{id}"] & & & &
\end{tikzcd}
$$

In particular the solid diagram is commutative with cocartesian squares. We have that $\iota \in \mathrm{Def}_A(S)$ maps to $\mathrm{id}\,\tilde{\otimes}1 \in \mathrm{Def}_A(R)$. Since ξ_v and ξ_0 by $\mathrm{Def}_{A\to k} \to \mathrm{Def}_A$ maps to $\mathrm{id}\,\tilde{\otimes}1$ and ι_0 respectively, it follows that ι maps to ι_0. By versality of ι there exists a lifting θ_1 of θ_0 such that ι maps to ι_1 in $\mathrm{Def}_A(S_1)$. The obtained map τ_1 lifts τ_0. Define α_1 as $\pi_1\tau_1$. Then α_1 lifts α_0 since $p\alpha_1 = p\pi_1\tau_1 = \pi_0 q\tau_1 = \pi_0\tau_0 = \alpha_0\mu(1\tilde{\otimes}\,\mathrm{id}) = \alpha_0$. Also $S_1\tilde{\otimes}_R R^{(2)} \cong S_1\tilde{\otimes}_S R \cong T_1$. Let β_1 be the induced map $R^{(2)} \to T_1$. It lifts β_0 and we get that $\mathrm{id}\,\tilde{\otimes}1$ maps to ι_1. But also the right square commutes since $(\pi_1\beta_1 - \alpha_1\mu)(1\tilde{\otimes}\,\mathrm{id}) = \pi_1\tau_1 - \alpha_1 = 0$ and $(\pi_1\beta_1 - \alpha_1\mu)(\mathrm{id}\,\tilde{\otimes}1) = \pi_1\iota_1\alpha_1 - \alpha_1 = 0$. □

Lemma 6.6. *Let $h^{\mathrm{ft}}\colon S \to R^{\mathrm{ft}}$ be a finite type homomorphism of noetherian rings. Let M be an R^{ft}-module. Let R denote the henselisation of R^{ft} in a maximal ideal $\mathfrak{m}$.*

(a) *There are natural isomorphisms of André-Quillen cohomology*

$$\mathrm{H}^i(S, R, R\otimes M) \cong \mathrm{H}^i(S, R^{\mathrm{ft}}, M)\otimes_{R^{\mathrm{ft}}} R \quad \textit{for all } i\,.$$

Suppose in addition that M is finite, h^{ft} is flat, S local henselian and $S/\mathfrak{m}_S \cong R^{\mathrm{ft}}/\mathfrak{m} \cong k$. Let $k \to A^{\mathrm{ft}}$ denote the central fibre of h^{ft} and put $\mathfrak{m}_0 = \mathfrak{m}A^{\mathrm{ft}}$. Assume $\operatorname{Spec} A^{\mathrm{ft}} \setminus \{\mathfrak{m}_0\}$ is smooth over k.

(b) *For all $i > 0$ the André-Quillen cohomology $\mathrm{H}^i(S, R, R\otimes M)$ is finite as S-module and there is a natural $R^{\mathrm{ft}}_{\mathfrak{m}}$-isomorphism*

$$\mathrm{H}^i(S, R, R\otimes M) \cong \mathrm{H}^i(S, R^{\mathrm{ft}}, M)_{\mathfrak{m}}\,.$$

Proof. See the proof of Lemma 10.1 in [18]. □

We will use the following notation and assumptions. Let x denote a sequence of variables $x_1, \dots, x_m$ and f a regular sequence of elements $f_1, \dots, f_c$ contained in $(x)^2 \subset k[x]$ for a field k. Put $A^{\mathrm{ft}} = k[x]/(f)$ and assume the André-Quillen cohomology $\mathrm{H}^1(k, A^{\mathrm{ft}}, A^{\mathrm{ft}}) \cong (A^{\mathrm{ft}})^{\oplus c}/(\operatorname{im}\nabla(f))$ has support in (x) where $\nabla(f)$ equals $(\partial f_i/\partial x_j)$. Let $e_1, \dots, e_c$ be the standard generators in $k[x]^{\oplus c}$ and pick elements $g_1, \dots, g_N$ in $k[x]^{\oplus c}$ such that $\{e_1, \dots, e_c, g_1, \dots, g_N\}$ induce a k-basis for the finite dimensional $(A^{\mathrm{ft}})^{\oplus c}/(\operatorname{im}\nabla(f))$. Let $g_j^{(1)}, \dots, g_j^{(c)}$ denote the c projections of g_j in $k[x]$. Let $z = z_1, \dots, z_c$ and $t = t_1, \dots, t_N$ be new sets of variables and define $F_i(x,t) = f_i(x) + \sum_j t_j g_j^{(i)}$ in $k[x,t]$. Pick liftings in Λ of the coeffiecients of F_i to obtain a lifting $\tilde{F}_i$ in $\Lambda[x,t]$ of F_i. Put ${}^v\tilde{F}_i = \tilde{F}_i + z_i \in \Lambda[x,t,z]$. Finally, let $s = s_1, \dots, s_m$ be another set of variables and let $\tilde{F}(x,t) - \tilde{F}(s,t)$ denote the sequence $\tilde{F}_1(x,t) - \tilde{F}_1(s,t), \dots, \tilde{F}_c(x,t) - \tilde{F}_c(s,t)$ in $\Lambda[x,s,t]$.

Corollary 6.7. *Given these assumptions and notation, let A be the henselisation of $A^{\mathrm{ft}} = k[x]/(f)$ in the maximal ideal $(x)A^{\mathrm{ft}}$. Let $R \to T \to R$ be the henselisation of*

$$\Lambda[s,t] \to \Lambda[x,s,t]/(\tilde{F}(x,t) - \tilde{F}(s,t)) \xrightarrow{x \mapsto s} \Lambda[s,t]\,.$$

Then $R \to T \to R$ gives a formally versal element for the pointed deformation functor $\mathrm{Def}_{A\to k}\colon {}_{\Lambda}\mathsf{H}/k \to \mathsf{Sets}$. If Λ is an excellent ring this element is versal.

Proof. Let $\iota\colon S \to R$ be the henselisation of $\Lambda[t,z] \to \Lambda[x,t,z]/({}^{v}\tilde{F})$ which we claim gives a (formally) versal element for Def_A. By Theorem 6.5 $R \to R\tilde{\otimes}_S R \to R$ gives a (formally) versal element for $\mathrm{Def}_{A\to k}$. Note that $\Lambda[x,t,z]/({}^{v}\tilde{F}) \cong \Lambda[x,t]$ where $z_i \mapsto -\tilde{F}_i$. Consider the copy $\Lambda[t,z] \to \Lambda[s,t]$ where $z_i \mapsto -\tilde{F}_i(s,t)$. There is a ring isomorphism $\Lambda[s,t]\otimes_{\Lambda[t,z]}\Lambda[x,t] \cong \Lambda[x,s,t]/(\tilde{F}(x,t) - \tilde{F}(s,t))$ and the corollary follows.

The claim is basically well known. We only sketch the argument. As in Example 2.3 $({}^{v}\tilde{F})$ is a regular sequence and R is S-flat. So ι is a deformation of A. To show formal versality put $\mathrm{H}^1 = \mathrm{H}^1(k,A,A)$ which is isomorphic to $\mathrm{H}^1(k, A^{\mathrm{ft}}, A^{\mathrm{ft}})$ by Lemma 6.6. Note that $S_{(1)} := S/(\mathfrak{m}_S^2 + \mathfrak{m}_\Lambda S) \cong k\oplus(\mathrm{H}^1)^*$. Now ι induces the universal deformation in $\mathrm{Def}_A(S_{(1)}) \cong \mathrm{End}_k(\mathrm{H}^1)$ corresponding to the identity. Given a lifting situation of elements in Def_A

(6.7.1)
$$\begin{tikzcd}
 & S^1 \arrow[rr, "\iota^1"] \arrow[d, "p"] & & R^1 \arrow[d, "q"] \\
 & S^0 \arrow[rr, "\iota^0"] & & R^0 \\
S \arrow[ur, "\theta^0"'] \arrow[uur, dotted, "\theta^1"] \arrow[rr, "\iota"] & & R \arrow[ur, "\tau^0"'] \arrow[uur, dotted, "\tau^1"] &
\end{tikzcd}$$

where the vertical maps are surjections, the solid squares are cocartesian and the S^i have finite length. We need to prove that a lifting θ^1 exists such that the third square is cocartesian and lifts the bottom square. By induction we assume $\mathfrak{m}_{S^1}\cdot \ker p = 0$. By picking elements $\theta(z_i)$ and $\theta(t_j)$ in S^1 lifting $\theta^0(z_i)$ and $\theta^0(t_j)$ we obtain a map $\theta\colon S \to S^1$. Let $R^2 = R\tilde{\otimes}_S S^1$. Then $\iota^2\colon S^1 \to R^2$ is in Def_A and lifts ι^0. By obstruction theory there is a transitive action of $\mathrm{H}^1(S^0, R^0, R^0\otimes \ker p)$ on the set of equivalence classes of liftings of ι^0 to S^1 [19, 2.1.3.3]. We have $R^0\otimes\ker p \cong A\otimes_k \ker p$. By [1, IV 54] we get

(6.7.2) $$\mathrm{H}^1(S^0, R^0, R^0\otimes\ker p) \cong \mathrm{H}^1(k,A,A)\otimes_k \ker p \cong \mathrm{Hom}((\mathrm{H}^1)^*, \ker p)\,.$$

Elements here can be ‘added’ to the ring homomorphism θ and adjusting by the difference of ι^1 and ι^2 gives a new ring homomorphism θ^1 inducing ι^1 from ι.

Since A^{ft} has an isolated singularity, [10, Théorème 8] gives that ι is versal. Here we use Artin's Approximation Theorem for an arbitrary excellent coefficient ring [2, 23, 24]; cf. R. Elkik's remark at the beginning of section 4 in [10]. □

For plane curve singularities the following explicit description of a stably reflexive complex E which is a hull for I in Theorem 6.1 implies that the obvious generalisation of Knudsen's stabilisation [20] has the relevant features.

Lemma 6.8. *Let $S \to P$ be a flat homomorphism of local noetherian rings and let $k = S/\mathfrak{m}_S$. Let P_0 denote $P\otimes_S k$ which we assume is a regular ring of dimension 2. Given an element $F \in \mathfrak{m}_P^2$ with image $f \neq 0$ in P_0. Put $R = P/(F)$ and suppose there is an S-algebra map $R \to S$. Let I_R denote the kernel.*

(i) *There are elements X_1, X_2, G_1, G_2 in $\mathfrak{m}_P$ with X_1 and X_2 inducing generators for I_R and satisfying $X_1G_1 + X_2G_2 = F$.*

(ii) *Given such elements, the* $(2 \text{ x } 2)$*-matrices*

$$\Phi = \begin{bmatrix} X_2 & G_1 \\ -X_1 & G_2 \end{bmatrix} \quad \textit{and} \quad \Psi = \begin{bmatrix} G_2 & -G_1 \\ X_1 & X_2 \end{bmatrix}$$

induces a 2-periodic R-complex

$$\dots \xrightarrow{\Psi} R^{\oplus 2} \xrightarrow{\Phi} R^{\oplus 2} \xrightarrow{\Psi} R^{\oplus 2} \xrightarrow{\Phi} R^{\oplus 2} \xrightarrow{\Psi} \dots$$

which is stably reflexive with respect to h and a hull for I_R.

Proof. (i) As in Example 2.3 R is S-flat. Let I_P be the kernel of the induced S-algebra map $P \to S$. Then $I_P \otimes_S k \cong \mathfrak{m}_{P_0}$ which is generated by two elements, say x_1 and x_2. Pick liftings X_i in I_P of the x_i. Then I_P is generated as ideal by X_1 and X_2. The kernel of $I_P \to I_R$ equals the kernel (F) of $P \to R$. In particular $F = X_1G_1 + X_2G_2$ for some elements G_1 and G_2 which have to be non-units.

(ii) Consider the commutative diagram with exact rows

(6.8.1)
$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r, "{[X_2,-X_1]^{\mathrm{tr}}}"] \arrow[d, "\tau"'] & P^{\oplus 2} \arrow[r, "{[X_1,X_2]}"] \arrow[d, equal] & I_P \arrow[r] \arrow[d, "\rho"] & 0 \\ 0 \arrow[r] & P^{\oplus 2} \arrow[r, "\Phi"] & P^{\oplus 2} \arrow[r] & \operatorname{coker}\Phi \arrow[r] & 0 \end{tikzcd}$$

The (inverse) connecting isomorphism $P = \operatorname{coker}\tau \cong \ker\rho$ takes 1 to $F \in I_P$. We get $\operatorname{coker}\Phi \cong I_P/(F) \cong I_R$. The pair (Φ, Ψ) is a matrix factorisation of F. Then $C(\Phi, \Psi)$ is a stably reflexive complex by Corollary 4.8 and a hull for I_R. □

With the notation and the assumptions from Lemma 6.8 we have:

Proposition 6.9. *Assume $P = S[X_1, X_2]_{\mathfrak{m}}$ where $\mathfrak{m} = \mathfrak{m}_S + (X_1, X_2)$ and let x_i and g_i denote the images in P_0 of X_i and G_i respectively. Suppose $g_i \in k[x_1, x_2]$. Then $\pi^{\mathrm{s}}\colon \operatorname{Proj}\operatorname{Sym}_R(I_R^\vee) \to \operatorname{Spec} S$ is flat with exceptional fibre E isomorphic to $\mathbb{P}^1_S$. The closed fibre of π^{s} has local complete intersection singularities and the section of π^{s} defined by the quotient $I_R^\vee \to I_R^\vee/R \cong S$ gives a smooth point in the closed fibre contained in $E \times_{\operatorname{Spec} S} \operatorname{Spec} k$.*

Proof. Since by Lemma 6.8 (i) $I_R^\vee \cong \operatorname{coker}\Phi^{\mathrm{tr}}$ we have

(6.9.1) $$\operatorname{Sym}_R(I_R^\vee) \cong R[U, V]/(X_2U + G_1V, G_2V - X_1U)\,.$$

Put $v = V/U$. Then $\mathcal{O}_{D_+(U)} \cong P[v]/(X_2 + G_1v, X_1 - G_2v)$ as F is contained in the ideal. Since the central fibre is obtained from $I_R^\vee \otimes_S k \cong (x_1, x_2)^\vee$ we can assume that $g_2 \in k[x_2]$ in the presentation matrix and in the presentation $\mathcal{O}_{D_+(U)} \otimes_S k \cong P_0[v]/(x_2 + g_1v, x_1 - g_2v)$ which then is isomorphic to $(P_0/(x_1))[v]/(x_2 + vg_1')$ where $g_1' = g_1(g_2v, x_2)$. It follows that $\mathcal{O}_{D_+(U)}$ is S-flat (see Example 2.3). Put $u = U/V$. Then $\mathcal{O}_{D_+(V)} \cong P[u]/(X_2u + G_1, X_1u - G_2)$ with central fibre $P_0[u]/(x_2u + g_1, x_1u - g_2)$ where we can assume $g_2 \in k[x_2]$. One checks that $x_1u - g_2$ is irreducible in $k[x_1, x_2, u]$. As in Example 2.3 $\mathcal{O}_{D_+(V)}$ is S-flat. Moreover, the closed fibre of π^{s} has local complete intersection singularities. The exceptional fibre is given by $\operatorname{Proj}\operatorname{Sym}_R I_R^\vee \otimes_R R/I_R$. By Theorem 6.1 (iv) $I_R^\vee \otimes_R R/I_R \cong S^{\oplus 2}$ as R-modules. By inspecting the presentations we find that the image of 1 in $R \to I_R^\vee$ is given by $[X_1, X_2]^{\mathrm{tr}}$. The section of π^{s} is therefore given by dividing the homogeneous coordinate ring out by V which gives the quotient ring $R[U]/(U)(X_1, X_2)$. □

7. The stabilisation morphism

Let $\bar{\mathscr{M}}_{g,n}$ denote the stack of stable n-pointed curves. An object in $\bar{\mathscr{M}}_{g,n}$ is a proper and flat map $\pi\colon C \to T$ of schemes together with n sections $\sigma_i\colon T \to C$ such that the geometric fibres $C_{\bar{t}}$ of π are curves which together with the points $\sigma_i(\bar{t})$ have certain properties; see [20]. The morphisms are commutative, cartesian

diagrams. There is also a stack $\bar{\mathscr{C}}_{g,n}$ with objects stable n-pointed curves plus an extra section $\Delta\colon T \to C$ without conditions. Forgetting this extra section gives a functor $\bar{\mathscr{C}}_{g,n} \to \bar{\mathscr{M}}_{g,n}$ and $\bar{\mathscr{C}}_{g,n}$ is called the universal curve. The stabilisation map is a functor $s\colon \bar{\mathscr{C}}_{g,n} \to \bar{\mathscr{M}}_{g,n+1}$. Knudsen uses it to study divisors on the $\bar{\mathscr{M}}_{g,n}$ and to construct the clutching morphisms. We sketch the main arguments in the construction of $s(\pi, \{\sigma_i\}, \Delta) = (\pi^s\colon C^s \to T, \{\sigma_i^s\})$. We assume that all schemes are noetherian.

Since the points of $\sigma_i(T)$ in C are smooth over T by [14, 6.7.8], the $\sigma_i(T)$ are locally principal divisors of C by [15, 17.10.4], but this is not the case for Δ. Put $\mathcal{I} = \mathcal{I}_{\Delta(T)}$. Let $\mathcal{O}_C \to \mathcal{O}_C(\Sigma_i\sigma_i)$ and $\mathcal{O}_C \to \mathcal{I}^\vee$ be the duals of the ideal inclusions. Let the coherent sheaf $\mathcal{K}$ on C be defined by the exact sequence

$$0 \to \mathcal{O}_C \longrightarrow \mathcal{I}^\vee \oplus \mathcal{O}_C(\Sigma_i\sigma_i) \xrightarrow{\rho} \mathcal{K} \to 0\,. \tag{7.0.2}$$

Let $q\colon C^s := \operatorname{Proj}\operatorname{Sym}_{\mathcal{O}_C}\mathcal{K} \to C$ and $\pi^s = \pi q$. The quotient $\mathcal{K} \to \mathcal{K}/\rho(\mathcal{O}_C(\Sigma_i\sigma_i)) = \mathcal{L}_\Delta$ defines a section $\Delta^s\colon T \to C^s$ lifting Δ provided we know that $\Delta^*\mathcal{L}_\Delta$ is a line bundle on T; cf. [13, 4.2.3]. This local question is answered in Corollary 6.3 (ii). Let $\sigma_{n+1}^s := \Delta^s$. Similarly $\sigma_i^*(\mathcal{K}/\rho(\mathcal{I}^\vee))$ defines the lifting σ_i^s for $i = 1, \dots, n$. If (7.0.2) commutes with base change s will define a functor (after choosing representatives which by construction are unique up to unique isomorphisms). Since there is a global comparison map $f^*(\mathcal{K}) \to \mathcal{K}'$ where $f\colon C' \to C$ is induced by base change, the question is local. The critical case is where $\Delta(\bar{t})$ is an ordinary double point (a node). After localisation $\mathcal{K}$ equals $\mathcal{I}^\vee$ and $f^*(\mathcal{I}^\vee) \to (f^*\mathcal{I})^\vee$ is an isomorphism by Corollary 6.3 (i).

To prove flatness of $C^s \to T$, we consider the local situation. Put $t = \pi(x)$ for $x \in C$, let $\mathcal{O}$ and Λ denote the henselisations of $\mathcal{O}_{C,x}$ and $\mathcal{O}_{T,t}$ respectively. Moreover, let $k = k(t)$ and $A = \mathcal{O}\otimes_T k$ with k-algebra map $A \to k$ given by the section $\Delta_{C,x}\otimes k$. Knudsen's idea is to write up an explicit (formally) versal family for $\operatorname{Def}_{A\to k}$, say with base ring R, do a local version of the construction of C^s over this family and inspect it for flatness. By versality (assuming Λ is excellent) $\Lambda \to \mathcal{O}$ is obtained by base change along some Λ-algebra map $R \to \Lambda$. This family is then flat and flatness of C^s along the fibre over x follows by faithful flatness of henselisation. The non-excellent case is similar, formal versality is sufficient, and one can even work with complete rings and formal families.

For the explicit formally versal (f.v.) family we consider the critical case where $\Delta(\bar{t})$ is an ordinary double point. In [21] Knudsen shows that $\hat{\mathcal{O}}\otimes_\Lambda k(t)$ is isomorphic to $k(t)[[x_1, x_2]]/(q)$ where $q = q(x) = x_1^2 + \gamma x_1x_2 + \delta x_2^2$ for γ and δ in $k = k(t)$ with discriminant $\gamma^2 - 4\delta \neq 0$. One could instead take the strict henselisation (also faithfully flat) of the local rings $\mathcal{O}_{T,t}$ and $\mathcal{O}_{C,x}$ and q would split as a product of two distinct linear forms over k^{sep}, but this simplification of the equation would only change the following argument nominally. Let A be the henselisation of $k[x_1, x_2]/(q)$. By Theorem 6.5 one obtains the f.v. family of $\operatorname{Def}_{A\to k}$ from the f.v. family of Def_A. The Zariski tangent space $\operatorname{Def}_A(k[\varepsilon])$ equals the first André-Quillen cohomology $\mathrm{H}^1(k, A, A) \cong A/(\operatorname{im}\nabla(q))$ where $\nabla(q)$ is the Jacobi matrix. By the discriminant condition

$$\operatorname{Def}_A(k[\varepsilon]) \cong k[x_1, x_2]/(q, 2x_1 + \gamma x_2, 2\delta x_2 + \gamma x_1) \cong k. \tag{7.0.3}$$

Put $\tilde{q} = \tilde{q}(x) = x_1^2 + \tilde{\gamma}x_1x_2 + \tilde{\delta}x_2^2$ where $\tilde{\gamma}$ and $\tilde{\delta}$ in Λ are liftings of γ and δ. Put ${}^v\tilde{Q} = \tilde{q} + z$ for a variable z and let $S \to R$ be the henselisation of $\Lambda[z] \to \Lambda[x_1, x_2, z]/({}^v\tilde{Q})$. This is a f.v. family for Def_A (cf. proof of Corollary 6.7). Note that $\Lambda[x_1, x_2, z]/({}^v\tilde{Q}) \cong \Lambda[x_1, x_2]$ where $z \mapsto -\tilde{q}(x)$. Corollary 6.7 shows that the f.v. family $R \to R^{(2)} \to R$ in $\operatorname{Def}_{A\to k}$ is particularly simple in this case. We use

s_1, s_2 instead of the variables x_1, x_2 in the left R. The obvious ring homomorphism

$$L := \Lambda[s_1, s_2, x_1, x_2]/(\tilde{q}(x) - \tilde{q}(s)) \longrightarrow \Lambda[s_1, s_2]\otimes_{\Lambda[z]}\Lambda[x_1, x_2] \tag{7.0.4}$$

is an isomorphism, $L^{\mathrm{h}} = R^{(2)}$ and $\hat{L}$ is precisely Knudsen's hull in Proposition 2.1 in [21].

The equations for C^s above $x \in C$ (after completion) is given in Proposition 6.9. Let ${}^{\mathrm{v}}S$ and ${}^{\mathrm{v}}P$ denote the Zariski localisation in the obvious maximal ideals of $\Lambda[s_1, s_2]$ and ${}^{\mathrm{v}}S[x_1, x_2]$, respectively. Put $F = \tilde{q}(x) - \tilde{q}(s)$ and ${}^{\mathrm{v}}R = {}^{\mathrm{v}}P/(F)$. Let I be the kernel of the ${}^{\mathrm{v}}S$-algebra map ${}^{\mathrm{v}}R \to {}^{\mathrm{v}}S$ defined by $x_i \mapsto s_i$ for $i = 1, 2$. To find the differential in the stably reflexive complex which is a hull for I (see Lemma 6.8), put $X_i = x_i - s_i$ for $i = 1, 2$. Then $G_1 = x_1 + \tilde{\gamma}x_2 + s_1$ and $G_2 = \tilde{\delta}(x_2 + s_2) + \tilde{\gamma}s_1$ gives a solution to the equation $F = X_1G_1 + X_2G_2$. The homogeneous coordinate ring is given in (6.9.1). In the closed fibre the local charts are (without localisation in (x)):

$$\mathcal{O}_{D_+(U)}\otimes k \cong k[x_2, v]/(x_2(\delta v^2 + \gamma v + 1)) \tag{7.0.5}$$

$$\mathcal{O}_{D_+(V)}\otimes k \cong k[x_2, u]/(x_2(u^2 + \gamma u + \delta)) \tag{7.0.6}$$

In particular these are local complete intersections and $\operatorname{Proj}\operatorname{Sym}_{{}^{\mathrm{v}}R} I^{\vee}$ is flat over ${}^{\mathrm{v}}S$. Moreover, the closed fibre is reduced and connected, the exceptional component $x_1 = 0 = x_2$ is a $\mathbb{P}^1_k$, and the intersection points with the other components in the geometric fibre are two distinct ordinary double points as the discriminants are non-zero. Also note that the image of 1 in ${}^{\mathrm{v}}R \to I^{\vee}$ is given by $[X_1, X_2]^{\mathrm{tr}}$ and the section Δ^s, given by the quotient $I^{\vee}/\,{}^{\mathrm{v}}R$, therefore is given by dividing out by the homogeneous coordinate V. In the local chart (7.0.5) this corresponds to $v = 0$ which is never a solution to $\delta v^2 + \gamma v + 1 = 0$.

Note that while Corollary 6.3 (i) and (ii) are used in this argument, Corollary 6.3 (iii) isn't. We use Theorem 6.1 (iv) in the proof of Proposition 6.9 and Knudsen applies Corollary 6.3 (iv) in the construction of the clutching morphisms; see [20, 3.7].

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